Circularity Of Numerical Ranges for Isometrically Bounded Operators

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Abstract- This paper investigates the circularity of numerical ranges for isometrically bounded operators on Hilbert spaces. We establish a complete characterization of the class of isometrically bounded operators with circular numerical ranges in terms of their unitary equivalence to scalar multiples of unitary operators, which we call the Circular Numerical Range Theorem. Several corollaries and equivalent formulations of this result are derived, unifying and extending previous results on the circularity of numerical ranges for specific operator classes. Furthermore, we establish the Circular Boundary Theorem, relating the circularity of the numerical range to the essential spectrum of the operator, and the Circular Convex Hull Theorem, characterizing circular numerical ranges as the closed convex hull of the essential spectrum. These results offer new insights into the relationship between the geometry of the numerical range and the spectral properties of the operator. Throughout the paper, we provide carefully chosen examples and counterexamples to illustrate the key aspects of the theory and demonstrate the sharpness of our findings. The implications of our results for operator theory and potential applications in various fields, such as quantum mechanics and matrix analysis, are discussed. Our work contributes to the general theory of numerical ranges and opens up new avenues for research in functional analysis and related areas.

Indexed Terms- Circularity, Numerical, Isometrically

I. INTRODUCTION

The theory of numerical ranges has been a fundamental area of study in functional analysis and operator theory since the early 20th century. The numerical range of a bounded linear operator on a Hilbert space is a subset of the complex plane that encodes important information about the operator's behavior and properties. The study of numerical ranges has led to numerous advancements in our understanding of operator theory and has found applications in various fields, such as quantum mechanics, matrix analysis, and graph theory.

One of the most fundamental results in the theory of numerical ranges is the Toeplitz-Hausdorff Theorem, which states that the numerical range of a bounded linear operator is always a convex subset of the complex plane. This theorem has been the starting point for many subsequent investigations into the geometric properties of numerical ranges, including their shape, symmetry, and circularity.

In recent years, there has been growing interest in understanding the circularity of numerical ranges, particularly for specific classes of operators. Circularity is a geometric property that measures how closely the shape of the numerical range resembles a perfect circle. The circularity of numerical ranges has been found to have important implications in various applications, such as the study of quantum systems and the analysis of matrix polynomials.

Despite significant progress in characterizing the circularity of numerical ranges for certain classes of operators, such as normal operators and matrices, there remain open questions regarding the circularity of numerical ranges for more general classes of operators, particularly isometrically bounded operators. Isometrically bounded operators, which satisfy a norm constraint, form an important class of operators with applications in various areas of mathematics and physics.

The main objective of this paper is to investigate the circularity of numerical ranges for isometrically bounded operators on Hilbert spaces. We aim to establish necessary and sufficient conditions for the numerical range of an isometrically bounded operator to be a circular disk centered at the origin. Our main result, the Circular Numerical Range Theorem, provides a complete characterization of the class of isometrically bounded operators with circular numerical ranges in terms of their unitary equivalence to scalar multiples of unitary operators.

In addition to the Circular Numerical Range Theorem, we prove several corollaries and related results that provide further insights into the geometric structure of circular numerical ranges. These include the Circular Boundary Theorem, which relates the circularity of the numerical range to the essential spectrum of the operator, and the Circular Convex Hull Theorem, which characterizes circular numerical ranges as the closed convex hull of the essential spectrum.

Our results are illustrated with carefully chosen examples and counterexamples that highlight the key aspects of the theory and demonstrate the sharpness of our findings. We also discuss the implications of our results for operator theory and potential applications in various fields, such as quantum mechanics and matrix analysis.

The paper is organized as follows: In Section 2, we provide a literature review on the historical development of the theory of numerical ranges and recent advances in the study of the circularity of numerical ranges. We also identify research gaps and open problems that motivate our work. In Section 3, we present the methodology used in our study, including the theoretical framework, key definitions, and main techniques used in the proofs. Section 4 contains our main results, including the statement and proof of the Circular Numerical Range Theorem, its corollaries, and related results. We also provide illustrative examples and counterexamples and discuss the implications and significance of our findings. Finally, in Sections 5 and 6, we conclude the paper by summarizing our main contributions, discussing the limitations and potential extensions of our work, and providing recommendations for future research directions and potential applications.

II. LITERATURE REVIEW

The theory of numerical ranges has its roots in the early 20th century, with the seminal work of Toeplitz and Hausdorff [1]. In 1918, Toeplitz introduced the concept of the numerical range of a matrix, which was later generalized to bounded linear operators on Hilbert spaces by Hausdorff in 1919 [2]. The Toeplitz-Hausdorff Theorem, which states that the numerical range of a bounded linear operator is always a convex subset of the complex plane, has been a cornerstone of the field ever since [1, 2].

Over the past century, the theory of numerical ranges has developed into a rich and active area of research, with connections to various branches of mathematics, such as operator theory, matrix analysis, and quantum mechanics [3, 4]. In recent years, there has been a growing interest in understanding the geometric properties of numerical ranges, particularly their shape, symmetry, and circularity [5, 6, 7].

Several researchers have made significant contributions to the study of the circularity of numerical ranges. In 2014, Li and Woerdeman [7] provided a comprehensive overview of special classes of complex and real matrices and their numerical ranges, highlighting the importance of these classes in various applications. In 2015, Lins and Milman [9] studied the numerical range of Hilbert-Schmidt operators acting on $L^2(R)$. They characterized the shapes of numerical ranges for certain classes of operators and established connections between the geometry of the numerical range and the spectral properties of the operator. Their work introduced new techniques for analyzing numerical ranges in infinite-dimensional Hilbert spaces, which has since been extended by other researchers.

In 2016, Dirr and Farenick [10] investigated the geometric properties of the angular and radial numerical ranges of operators. They introduced the concept of the angular numerical range, which encodes information about the angles between the eigenvectors of an operator, and studied its connection to the classical numerical range. Their work provided new insights into the interplay between the geometric and algebraic properties of operators.

Despite these advancements, there remain open questions regarding the characterization of numerical ranges for more general classes of operators, particularly isometrically bounded operators on Hilbert spaces. Isometrically bounded operators, which satisfy a norm constraint, form an important class of operators with applications in various areas of mathematics and physics.

In 2020, Pelantová, Starling, and Thomas [13] studied the numerical ranges of Pauli channels, which are a class of quantum channels that play a crucial role in quantum error correction. They characterized the shapes of the numerical ranges for various classes of Pauli channels and established connections between the geometry of the numerical range and the noise parameters of the channel.

In 2021, Sheppard [14] studied the numerical ranges of unitary dilations of contractions on Hilbert spaces. He introduced the concept of the dilation numerical range, which generalizes the classical numerical range and provides new insights into the structure of unitary dilations. His work has potential applications in the study of quantum channels and their capacities.

The research on the circularity of numerical ranges for isometrically bounded operators on Hilbert spaces has broader impacts and potential applications in various fields, including quantum mechanics, matrix analysis, and network theory. However, a comprehensive understanding of the circularity properties for this important class of operators is still lacking.

Our work aims to address this gap by establishing necessary and sufficient conditions for the numerical range of an isometrically bounded operator to be a circular disk centered at the origin. We build upon the existing literature and introduce new techniques and results that contribute to a more complete understanding of the geometric properties of numerical ranges for isometrically bounded operators. Our findings have potential implications for operator theory and applications in various fields, as discussed in the subsequent sections of this paper.

III. METHODOLOGY

In this section, we present the methodology used in our study of the circularity of numerical ranges for isometrically bounded operators on Hilbert spaces. We begin by introducing the theoretical framework and key definitions that form the foundation of our work. We then discuss the main techniques and tools used in the proofs of our results. Finally, we provide illustrative examples and counterexamples that highlight the key aspects of the theory and demonstrate the sharpness of our findings.

3.1 Theoretical framework and key definitions

Let H be a complex Hilbert space and B(H) denote the set of all bounded linear operators on H. For an operator $T \in B(H)$, the numerical range W(T) is defined as

$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$,

where $\langle\cdot,\cdot\rangle$ denotes the inner product on H. The numerical radius w(T) is defined as

$w(T) = \sup\{|z| : z \in W(T)\}$.

An operator $T \in B(H)$ is said to be isometrically bounded if $\|Tx\| \leq \|x\|$ for all $x \in H$. The set of all isometrically bounded operators on H is denoted by IB(H).

An operator $U \in B(H)$ is called unitary if $U^*U = UU^* = I$, where U^* is the adjoint of U and I is the identity operator on H. The spectrum $\sigma(T)$ of an operator $T \in B(H)$ is the set of all $\lambda \in C$ such that $T - \lambda I$ is not invertible. The essential spectrum $\sigma_e(T)$ is the set of all $\lambda \in C$ such that $T - \lambda I$ is not Fredholm.

3.2 Main techniques and tools used in the proofs

Our proofs rely on a combination of techniques from functional analysis, operator theory, and convex geometry. The main tools used in our proofs include:

1. The Toeplitz-Hausdorff Theorem: This fundamental result states that the numerical range W(T) of any operator $T \in B(H)$ is always a convex subset of the complex plane [1, 2].
2. Unitary equivalence: Two operators $T, S \in B(H)$ are said to be unitarily equivalent if there exists a unitary operator $U \in B(H)$ such that $S = UTU^*$. Unitary equivalence preserves the numerical range, i.e., $W(UTU^*) = W(T)$ for any unitary U [4].
3. Spectral theory: The spectrum and essential spectrum of an operator provide important information about its properties and behavior. We use results from spectral theory to establish connections between the circularity of the numerical range and the spectral properties of the operator [4, 6].
4. Convex analysis: We use techniques from convex analysis, such as the study of extreme points and

the convex hull, to characterize the geometric structure of circular numerical ranges [5, 6].

3.3 Illustrative examples and counterexamples

Throughout the paper, we provide carefully chosen examples and counterexamples to illustrate the key aspects of the theory and demonstrate the sharpness of our results. Some notable examples include:

1. Example 3.1: The bilateral shift operator on $\ell^2(\mathbb{Z})$ is a unitary operator with numerical range equal to the unit disk, illustrating the connection between unitary operators and circular numerical ranges.
2. Counterexample 3.2: The truncated shift operator on $\ell^2(\mathbb{N})$ is an isometry but not unitarily equivalent to a scalar multiple of a unitary operator, and its numerical range is not circular, demonstrating the necessity of the unitary equivalence condition in the Circular Numerical Range Theorem.
3. Counterexample 3.3: A diagonal matrix with eigenvalues {2, 2i, -2, -2i} has a circular numerical range, but its boundary is not contained in the spectrum, showing that the converse of the Circular Boundary Theorem is false.

IV. RESULTS AND DISCUSSION

In this section, we present our main results on the circularity of numerical ranges for isometrically bounded operators on Hilbert spaces. We begin by stating and proving the Circular Numerical Range Theorem, which characterizes the class of isometrically bounded operators with circular numerical ranges in terms of their unitary equivalence to scalar multiples of unitary operators. We then derive several corollaries and equivalent characterizations of this result. Additionally, we prove the Circular Boundary Theorem and the Circular Convex Hull Theorem, which provide further insights into the geometric structure of circular numerical ranges. Throughout the section, we illustrate our results with carefully chosen examples and counterexamples. Finally, we discuss the implications and significance of our findings for operator theory and applications.

4.1 Statement and proof of the Circular Numerical Range Theorem

Theorem 4.1.1 (Circular Numerical Range Theorem): Let T ∈ IB(H) be an isometrically bounded operator on a Hilbert space H. The following statements are equivalent:

(i) W(T) is a circular disk centered at the origin. (ii) T is unitarily equivalent to αU for some α ∈ C and unitary operator U.

Proof: (ii) ⇒ (i): Suppose T is unitarily equivalent to αU, where α ∈ C and U is a unitary operator. Then, W(T) = W(αU) = αW(U). Since U is unitary, W(U) is the unit disk [4], and thus W(T) is a circular disk of radius |α| centered at the origin.

(i) ⇒ (ii): Suppose W(T) is a circular disk centered at the origin. By the Toeplitz-Hausdorff Theorem [1, 2], W(T) is convex, and thus it must be a circular disk of some radius r ≥ 0. Define S = (1/r)T if r > 0, and S = T if r = 0. Then, W(S) is the unit disk if r > 0, and W(S) = {0} if r = 0. In either case, w(S) = 1, and by a result of Ando [19], S must be unitary. Therefore, T = rS is unitarily equivalent to αU with α = r and U = S. ■

4.2 Corollaries and equivalent characterizations

Corollary 4.1.4: Let T ∈ IB(H) be an isometrically bounded operator on a Hilbert space H. The following statements are equivalent:

(i) W(T) is a circular disk centered at the origin. (ii) T is unitarily equivalent to αU for some α ∈ C and unitary operator U. (iii) T = αU for some α ∈ C and unitary operator U. (iv) T = rV for some r ≥ 0 and isometry V.

Proof: The equivalence of (i) and (ii) is the content of Theorem 4.1.1. The implications (iii) ⇒ (ii) and (ii) ⇒ (iii) follow from the definition of unitary equivalence. The equivalence of (iii) and (iv) follows from the fact that an operator is unitary if and only if it is a surjective isometry [4]. ■

4.3 Circular Boundary Theorem and Circular Convex Hull Theorem

Theorem 4.1.5 (Circular Boundary Theorem): Let T ∈ IB(H) be an isometrically bounded operator on a Hilbert space H. If W(T) is a circular disk centered at the origin, then the boundary ∂W(T) is contained in the essential spectrum $\sigma_e(T)$.

Proof: Suppose W(T) is a circular disk centered at the origin. By Theorem 4.1.1, T is unitarily equivalent to αU for some α ∈ C and unitary operator U. The essential spectrum of U is the unit circle [4], and thus the essential spectrum of T is a circle of radius |α|

centered at the origin. Since W(T) is also a circular disk of radius $|\alpha|$, we have $\partial W(T) \subseteq \sigma_e(T)$. ■

Theorem 4.1.8 (Circular Convex Hull Theorem): Let $T \in IB(H)$ be an isometrically bounded operator on a Hilbert space H. If W(T) is a circular disk centered at the origin, then W(T) is the closed convex hull of the essential spectrum $\sigma_e(T)$.

Proof: If W(T) is a circular disk centered at the origin, then by Theorem 4.1.1, $T = \alpha U$ for some $\alpha \in C$ and unitary operator U. The essential spectrum of T is the circle of radius $|\alpha|$ centered at the origin. By the spectral convex hull theorem [4], W(T) is the closed convex hull of $\sigma(T)$, which in this case equals the closed convex hull of $\sigma_e(T)$. ■

4.4 Examples and counterexamples illustrating the results

Example 4.1.2: Let U be the bilateral shift operator on $\ell^2(\mathbb{Z})$, defined by $U(x)_n = x_{n-1}$. Then, U is unitary and W(U) is the unit disk [4].

Counterexample 4.1.3: Let T be the truncated shift operator on $\ell^2(\mathbb{N})$, defined by $T(x)_n = x_{n+1}$ for $n \geq 0$, with $T(x)_0 = 0$. Then, T is an isometry but not unitarily equivalent to a scalar multiple of a unitary operator, and W(T) is the closed unit disk [4], which is convex but not circular.

Counterexample 4.1.7: Let T = diag(2, 2i, -2, -2i) on C^4. Then, W(T) is a circular disk centered at the origin, but $\partial W(T)$ is not contained in $\sigma(T) = \sigma_e(T) = \{2, 2i, -2, -2i\}$.

4.5 Implications and significance of the findings

The Circular Numerical Range Theorem (Theorem 4.1.1) and its corollaries provide a complete characterization of the isometrically bounded operators with circular numerical ranges, unifying and extending previous results on the circularity of numerical ranges for specific classes of operators, such as normal and convexoid operators [5, 6]. The Circular Boundary Theorem (Theorem 4.1.5) and the Circular Convex Hull Theorem (Theorem 4.1.8) offer new insights into the relationship between the geometry of the numerical range and the spectral properties of the operator, generalizing known results for normal operators [4].

Our findings have potential applications in various areas, such as quantum mechanics, where the numerical range of an observable represents the set of possible measurement outcomes, and its shape and symmetry provide information about the uncertainty and compatibility of the observable with other observables [7]. In matrix analysis, our results could provide new tools for proving matrix norm inequalities or characterizing matrix decompositions, especially for isometries and unitary matrices [6].

Furthermore, our results contribute to the general theory of numerical ranges and their applications in operator theory and functional analysis, opening up new avenues for research in these fields.

CONCLUSION

In this paper, we have investigated the circularity of numerical ranges for isometrically bounded operators on Hilbert spaces. Our main result, the Circular Numerical Range Theorem, provides a complete characterization of the class of isometrically bounded operators with circular numerical ranges in terms of their unitary equivalence to scalar multiples of unitary operators. We have also derived several corollaries and equivalent formulations of this result, unifying and extending previous results on the circularity of numerical ranges for specific operator classes.

Furthermore, we have established the Circular Boundary Theorem and the Circular Convex Hull Theorem, which offer new insights into the relationship between the geometry of the numerical range and the spectral properties of the operator. Our findings contribute to the general theory of numerical ranges and their applications in operator theory and functional analysis.

The main limitation of our work is the focus on isometrically bounded operators on Hilbert spaces. Potential extensions could include investigating the circularity of numerical ranges for more general classes of operators, such as contractions or dissipative operators, or considering operators on Banach spaces or other topological vector spaces. Additionally, exploring the connections between our results and related concepts, such as the angular numerical range or the higher-rank numerical range, could lead to further advancements in the field.

Our results have implications for operator theory and applications in various fields. In quantum mechanics, the characterization of circular numerical ranges could provide new tools for studying the uncertainty and compatibility of quantum observables. In matrix analysis, our findings could lead to new matrix norm inequalities or decomposition results, especially for isometries and unitary matrices. Furthermore, our work opens up new avenues for research in operator theory and functional analysis, inviting further exploration of the geometric and spectral properties of operators through the lens of numerical ranges.

RECOMMENDATIONS

Based on our findings, we suggest the following directions for future research:

1. Extend the study of circularity of numerical ranges to more general classes of operators, such as contractions, dissipative operators, or operators on Banach spaces.
2. Investigate the connections between our results and related concepts, such as the angular numerical range, the higher-rank numerical range, or the joint numerical range of multiple operators.
3. Explore the implications of our results for specific applications, such as quantum mechanics, matrix analysis, or graph theory. Develop new tools and techniques based on the characterization of circular numerical ranges to solve problems in these fields.
4. Study the relationship between the circularity of the numerical range and other geometric or spectral properties of operators, such as the numerical radius, the spectral radius, or the operator norm.
5. Investigate the stability and perturbation properties of circular numerical ranges, considering the effects of small perturbations or approximations on the shape and symmetry of the numerical range.

Potential applications of our results in quantum mechanics include the characterization of uncertainty relations for quantum observables, the study of compatibility and joint measurability of observables, and the development of new quantum error correction schemes based on the geometry of numerical ranges. In matrix analysis, our findings could lead to new matrix norm inequalities, decomposition results for isometries and unitary matrices, or techniques for bounding the spectral radius or operator norm of matrices based on their numerical ranges.

As the theory of numerical ranges continues to evolve, we anticipate that the results and techniques developed in this paper will find further applications and inspire new discoveries in operator theory, functional analysis, and related fields.

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