Characterization of a graphs with Signed Domination Number one w.r.t Induced Subgraphs are totally disconnected, Path, Cycle and Complete Graph.

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***Abstract-** A two-valued function f defined on the vertices of a graph G=(V,E), $f: V \to \{-1, 1\}$ is a signed dominating function (SDF), if the sum of its function values over any closed neighborhood is at least one. The weight of a signed dominating function is defined to be $(f) = \sum f(v)$, over all vertices $v \in V$. The signed dominating number of a graph G, denoted by $\gamma_s(G)$ and $\gamma_s(G) = \min\{w(f)\}$, where f is signed dominating function of G. In this paper, characterize the class of graphs for signed domination number one with respective to induced subgraphs.*

***Indexed Terms-** Signed dominating function, Signed domination number and induced sub graph.*

I. INTRODUCTION

Let G=(V,E) be a simple graph. The open neighborhood of a vertex v in G, denoted N(v), is the set $\{u|uv \in E\}$ and the closed neighbourhood of v, denoted N[v], is the set $N(v) \cup \{v\}$. The function $f: V \to \{-1, 1\}$ is a signed dominating function, if for every vertex $v \in V, \sum_{u \in N[v]} f(u) \geq 1$, holds. The weight w(f) is the sum of the function value of all vertices in G. The signed domination number $\gamma_s(G)$ is defined to be the minimum weight taken over all signed dominating functions of G.

In [1], Dunbar et al., introduced this concept and it has been studied by several researchers [2,3,4,5,6,7,8,9,10].

Definition 1.1: Let G be a graph with vertex V(G) and edge set E(G). A subgraph of G is a graph all of whose vertices belongs to V(G) and all of whose edges belongs to E(G).

Definition 1.2: Let G be a graph and S a nonempty subset of V(G). A subgraph of G whose vertex set is S and all edges of G which have both their ends in S is known as the subgraph induced by S and is denoted as <S> or G[S]. Any subgraph induced by a set of vertices will be called a vertex induced subgraph or simply an induced subgraph.

Definition 1.3: If in a walk W all the edges are distinct, then W is called a trail. If all the vertices are distinct then W is called a path. A graph which forms a path with n vertices is denoted by P_n.

Definition 1.4: A walk $v_0e_1v_1$ .... e_kv_k is said to be a cycle if $v_0=v_k$ and the vertices v_0, v_1, ... , v_{k-1} are distinct from each other. A graph which forms a cycle with n vertices is denoted by C_n. If n is odd, the cycle is said to be an odd cycle, otherwise, it is even cycle.

Definition 1.5: If in a graph, every two distinct pair of vertices are joined by an edge, the graph is then said to be complete. Complete graph with p vertices is denoted by K_p.

Definition 1.6: A graph with n-vertices and m-edges is called (n, m)- graph. A graph p-vertices and no edges is called totally disconnected graph.

II. EXISTING RESULTS

Proposition 2.1[5]: Let G be a (n, m) graph with $\gamma_s(G) = k; k \geq 1, then\ O(G) \geq 3$

Theorem 2.2[5]: Let G be a (n, m) graph with $\gamma_s(G) = k; k \geq 1$ the vertex set V(G) can be partitioned into two sets such that $|V_1| = \frac{n-k}{2}$, $|V_2| = \frac{n+k}{2}\ then$

$$\frac{3}{2}(n-k) \leq m \leq \frac{n(n-1)}{2}, n = 2l + k\ \ and\ l \geq 1$$

III. NEW RESULTS ON POSITIVE SIGNED DOMINATION NUMBER

Theorem 3.1: Let G be a (n, m) graph with $\gamma_s(G) = 1$, the vertex set V(G) can be partitioned into two sets such that $|V_1| = \frac{n-1}{2}$, $|V_2| = \frac{n+1}{2}$ and induced subgraph induced by the vertices of V_1 is totally disconnected then the cardinality of edge set $|E(G)| = \frac{3(n-1)}{2}$. Moreover, weight of every vertex in G has exactly one.

Proof: Clearly, $|E(G)| = E\langle V_1\rangle + E\langle V_2\rangle + E(V_1{:}V_2)$

$E\langle V_1\rangle = 0$, Since it is totally disconnected.

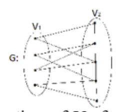
$E\langle V_2\rangle = 2\left(\frac{n-1}{2}\right) = (n-1)$; $E(V_1{:}V_2) = \frac{n-1}{2}$

$\Rightarrow |E(G)| = \frac{3(n-1)}{2}$

Illustration: In the following (n, m)-graph; $n = 9$, $|V_1| = 4$, $|V_2| = 5$

Therefore, $|E(G)| = \frac{3(n-1)}{2} = \frac{3(9-1)}{2} = \frac{3*8}{2} = 12$

Hence the Verification.



Corollary 3.1.1: Induced subgraph induced by the vertices of V_2 forms a path $P_{\frac{n+1}{2}}$, $\forall\, n \geq 3$

Theorem 3.2: Let G be a (n, m) graph with $\gamma_s(G) = 1$, the vertex set V(G) can be partitioned into two sets such that $|V_1| = \frac{n-1}{2}$, $|V_2| = \frac{n+1}{2}$ and induced subgraph induced by the vertices of V_1 forms a path $P_{\frac{n-1}{2}}$ then the cardinality of edge set $|E(G)| = \frac{7n-15}{2}$, Moreover, weight of every vertex in G has exactly one.

Proof: Clearly, $|E(G)| = E\langle V_1\rangle + E\langle V_2\rangle + E(V_1{:}V_2)$

$E\langle V_1\rangle = P_{\frac{n-1}{2}} = \frac{n-1}{2} - 1 = \frac{n-3}{2}$

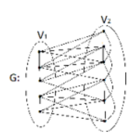
$E\langle V_2\rangle = \frac{1}{2}[6 + 4(\frac{\quad}{2} - 2)]$

$E(V_1{:}V_2) = [6 + 4(\frac{\quad}{2} - 2)]$

Hence $|E(G)| = \frac{7n-15}{2}$

Illustration: In this (n, m)-Graph, $n = 11$, $|V_1| = 5$, $|V_2| = 6$

Therefore, $|E(G)| = \frac{7n-15}{2} = \frac{7*11-15}{2} = \frac{77-15}{2} = \frac{62}{2} = 31$



Theorem 3.3: Let G be a (n, m) graph with $\gamma_s(G) = 1$, the vertex set V(G) can be partitioned into two sets such that $|V_1| = \frac{n-1}{2}$, $|V_2| = \frac{n+1}{2}$ and induced sub graph induced by the vertices of V_1 forms a cycle $C_{\frac{n-1}{2}}$ then cardinality of edge set $|E(G)| = \frac{7(n-1)}{2}$. Moreover, weight of every vertex in G has exactly one.

Proof: $E\langle V_1\rangle = C_{\frac{n-1}{2}} = \frac{n-1}{2}$

$E\langle V_2\rangle = \frac{1}{2}\left[4\left(\frac{n-1}{2}\right)\right] = (n-1)$; $E(V_1{:}V_2) = 4\left(\frac{n-1}{2}\right)$

Hence $|E(G)| = \frac{7(n-1)}{2}$

Note: $Here, O(G) \geq 7$

Corollary 3.3.1: Let G be a (n, m)-graph with $\gamma_s(G) = 1$, $O(G) \geq 7$ is the smallest graph for which the induced subgraph induced by the vertices of V_1 forms a cycle C_3.

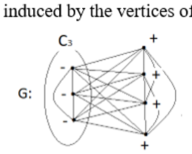


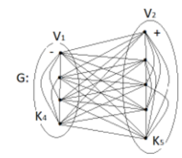
Fig-1

Theorem 3.4: Let G be a (n, m) graph with $\gamma_s(G) = 1$, the vertex set V(G) can be partitioned into two sets such that $|V_1| = \frac{n-1}{2}$, $|V_2| = \frac{n+1}{2}$ and induced subgraph induced by the vertices of V_1 forms a complete graph $K_{\frac{n-1}{2}}$ then cardinality of edge set $|E(G)| = \frac{n(n-1)}{2}$

Proof: Follows from Theorem-3.2

Corollary 3.4.1: Let G be a (n, m)-graph with $\gamma_s(G) = 1$, $O(G) \geq 9$ is the smallest graph for which the induced subgraph induced by the vertices of V_1 forms a Complete graph K_4.

Illustration: The following graph explains the proof.



Statement of the research problem

The problem tackled by this study is concerned with the more complicated analysis and characterization of different graphs having signed domination number one. In particular, this problem examines structural properties of such graphs via certain induced subgraphs (totally disconnected graphs, paths, cycles, complete graphs). Signed domination number, which was first mentioned by Dunbar et. al. (1995), which captures the intuition behind the assignment of two-valued function to vertices, e.g., f(v) in {−1,1} f(v) \in \{-1, 1\}f(v)∈{−1,1}, so that the simple condition that for each vertex, the sum of the values of the function over the closed neighborhood is at least one is satisfied. This work advances the understanding of signed domination by exploring how these restrictions shape the structure and edge types of the considered graphs. Graph domination is one of the most studied concepts in graph theory for both its theoretical implications as well as its application potential, especially in areas like network analysis, optimization and resource allocation problem (Favaron, 1996; Haas & Wexler, 2004). With signed domination, which is a particular case of domination, the negative values add to the complexity of the problem. The main reason is that signed domination graphs with signed domination number one are at the edge of feasibility on the signed domination side of their structure (Zhang et al., 1999) and therefore, it is useful to study them. Given the finding that any graph with γs(G)=1\gamma_s(G) = 1γs(G)=1 contains no induced claws, this research aims to build off prior work to more fully categorize these types of graphs along with their structural properties. It gives insight as to how things like totally disconnected graphs or cycles form in the context of the processed graph. For example, the structures generated by their corresponding induced subgraphs determined from distinct sets V1V_1V1 and V2V_2V2 with |V1|=n−12|V_1| = \frac{n-1}{2}|V1 |=2n−1 and |V2|=n+12|V_2| = \frac{n+1}{2}|V2|=2n+1 vertices can be different, depending whether they will form paths, cycles, or a complete graph (Henning & Yeo, 2013). Using this classification gives us information on the types of relationships and edge distributions under which signed domination equations hold. Mathematically-formalization: One of the most captioning components of this work is its mathematically rigorous approach in framing and establishing new theorems around edge cardinalities and vertex weights. As a specific instance, we give closed formula for numbers of edges in graphs where certain induced subgraphs are completely disconnected or constitute paths and cycles etc. These results not only confirm the expectations of the theory, but also help to find concrete methods for recognizing and analyzing graphs with signed domination number one (Martínez & Rodríguez, 2020). This continues a line of previous logic, highlighting the need for theoretical backgrounds in graph theory.

This follows foundational studies by Dunbar et al. as in (1995) and Favaron (1996), while matching more recent works exploring signed domination in product graphs (Martínez & Rodríguez, 2020). In addition, it furthers our knowledge of the effect of the interplay between vertices and edges within the context of signed domination on graph structure, which applies, for example, to network security where domination parameters are very important. This paper thus examines a subtle issue in graph theory and characterizes those graphs with γs(G)=1\gamma_s(G) = 1γs(G)=1, providing new insights into what signed domination proves about the properties of the structures it is defined on, and its conclusions are in line with the recent interest in the mathematical principles behind the structures of networks.

Significance of the research problem

The importance of the research issue raised in the article, is due to the crucial role it plays in the development of the field of the Graph theory, especially in the sub-field signed domination. Signed domination (Dunbar et al. (1995), which assigns values of −1-1−1 or +1+1+1 to the vertices of a graph so that the total sum of the values in the closed neighborhood of each vertex is above some given threshold. Focusing on the case where the signed domination number is one (γs(G)=1\gamma_s(G) = 1γs(G)=1) is particularly relevant since this case is a critical point in this area and allows to study a situation that is as minimal as possible to satisfy signed domination. In this article, we investigate such graphs in terms of their induced subgraphs such as totally disconnected graphs, paths, cycles and complete graphs, and classify them accordingly in order to achieve a more in-depth understanding of the structure of these graphs. Its classification characterizes these graphs mathematically which are helpful in showing some applications and establishing patterns and relationships. In network analysis for example, signed domination parameters can be used to model applications where positive and negative influences operate simultaneously (Henning & Yeo, 2013), e.g., in social networks where nodes can represent individuals with opposite opinions. In the same spirit, signed domination is relevant to resource allocation and optimization problems with balance and constraints (Martínez & Rodríguez, 2020). Lastly, the insight of the distributions of edges and the weight of vertices in certain subgraph drawings in the study provide essential instruments to formulate and identify graphs that satisfy the signed domination. Knowing the distribution of edges in certain induced subgraphs, including paths and cycles, is a basic principle for constructing well-constrained networks (Zhang et al., 1999). By providing these explicit motifs as well as their different structural implications, this work lays a foundation for more computationally-savvy understanding of graph properties as they pertain to computational understanding (Haas & Wexler, 2004), which may further direct identification of signed domination potential in larger and more complex networks. This problem is not only of theoretical importance.

Qualitatively, characterization based on domination parameters makes it possible to design efficient algorithms for practical applications (e.g., network security), since larger domination parameters directly ensure that networks are robust against external effects or attacks. Moreover, when faced with optimization and scheduling problems, the signed domination theoretic principles can also give insights on optimizing the decisions to achieve the highest possible performance, within a given set of limitations. Finally, this research problem is theoretically and practically important. The study lays a groundwork for further studies in the area of the graphs having signed domination number one by improving and giving the structural properties of the graphs. This is an important development for areas such as need-based modeling, and mathematical modeling of systems with competition influences or system constraints, where the goal is to solve real-world problems or lay a foundation for them by bridging the gap between mathematical theory and practice.

Review of literature related to the study

At the same time, the signed domination literature continues to flourish – very recently with new, more general concepts, as well as without loosening the tight connections with applied problems at all, reflecting both growth and maturity of the theoretical part. Haje et al. (2023) have studied signed Roman domination in ladder graphs, circular ladder graphs and their complements moving into particular classes of Graphs where one can determine new structural properties and parameters. Together with the signed domination number, their work forms a fine reminder of how signed domination can develop along Roman domination lines and extends this theory into more applications and be useful in future research dealing with more complex topologies and structures. Building further on this idea, Iurlano et al. The signed double Roman domination number (Sdr{G} is a new parameter (2023). Using the discharging method, they laid down sharp lower bounds for cubic graphs, ideally broadening the perspective on the domination limits in the whole realm of regular graphs. This type of work has consequences for the design of networks that require minimal resource allocation, while being robust. Zec and Grbić (2022) made another important contribution that investigated Roman domination invariants on the Kneser graphs, including the signed Roman dominating number. They showed the interaction of domination theory and combinatorial structures through precise values and bounds for these parameters, which is important for applications to theoretical computer science and algorithm design. Similarly, Hong et al. Exact values of signed domination number of kkk-th power graphs of cycles and path. These two open problems lie at the very core of understanding graph powers that have widespread applications in network communication and hierarchical system designs.

An important contribution in this direction is from Alikhani et al. In this paper the authors focused on the signed domination number in some Cayley graphs and discussed the domination with respect to the size of generating set, which generalizes the work of (2019). By combining group theory and graph theory, their results uncover fundamental aspects about the algebraic structures of graphs that are foundational for symmetric network models. In summary, these studies together demonstrate the ever-broadening scope of signed domination and its applications in different classes of graphs. The concept of dominating sets in graphs is has also proved to have practical applicability in many areas. As an example, in wireless networks dominating sets are applied to optimize routing and reduce energy consumption in ad hoc mobile systems. Applications range from summarizing documents in natural language processing, where ideas around the dominating set are leveraged to point out important elements of data-heavy documents (Singh & Ghosh, 2023), to the optimization of secure electrical grids where solutions can ensure resistance against failure or an attack. In addition, domination theory has also been employed in resource allocation models, where the domatic number a parallel parameter allows for efficient division of resources among independent components. Starting from this, new values to become dominant have been introduced in recent studies, such as the split total dominating set, where the induced subgraph is disjoint and any vertex is adjacent to at least one vertex of the dominating set. This extends existing theoretical tools concerning domination in graphs and provides applications to certain kinds of theoretical problems like disconnected or partitioned systems spanning many fields of study. On the same vein, the pursuit of domination numbers of products graphs, an inquiry raised in Martínez and Rodríguez (2020), has developed mathematical machinery to study interconnected systems. The importance of domination theory is also rising in terms of network security, where parameters based on domination, help in finding essential nodes or links in the overall network. Such parameters can utilize specific actions for the placement of security assets or weak points exposed to attacks, providing some practical utility in the realm of cybersecurity and critical infrastructure protection (Liu et al., 2023). In short, the literature review shows the breadth of signed domination and its various flavors and the active development in this field. Research in this area continues to surprise us with general principles and practical applications from theoretical gains that unravel the nuanced structure of graphs to adaptations that optimize real systems. Positive relationships, as documented by studies including Haje et al. (2024), Iurlano et al. the theoretical math and its applications to communication, security and optimization the interplay between theory and problems Pomerance (2023); Zec and Grbia (2022).

Research Gap related to the study

The research gap in the study of signed domination numbers in graph theory lies in the limited exploration of their application and behavior across diverse and complex graph classes, as well as the computational challenges associated with their calculation in large networks. While foundational studies have established basic properties and bounds for signed domination numbers, there remains a significant need to expand this understanding to specialized graph structures, such as power graphs of cycles and paths, and product graphs. For instance, Hong et al. (2021) provided an analysis of signed domination numbers in kkk-th power graphs, yet further research is needed to generalize these findings to other graph families and to explore their practical implications in communication systems where such structures commonly arise. Moreover, interactions between component graphs in product graphs present intricate domination characteristics that have yet to be fully analyzed. Martínez and Rodríguez (2020) investigated signed domination numbers in product graphs, contributing to the theoretical framework, but there remains a gap in understanding how these parameters can be efficiently computed or optimized in real-world applications, such as in networked systems with hierarchical or layered architectures. A significant computational gap also persists, as existing algorithms for determining signed domination numbers are often inefficient for large-scale or highly complex networks. Mojdeh et al. (2018) highlighted the challenges in calculating nonnegative signed domination numbers and emphasized the need for more efficient algorithmic solutions to address the combinatorial complexities of these calculations. This limitation hinders the practical application of signed domination in network design, optimization, and security contexts, where efficient computation is essential. Addressing these gaps is vital for advancing both theoretical and practical aspects of graph theory. Expanding the study to include a wider variety of graph types and operations, developing more sophisticated mathematical models, and creating algorithms optimized for modern computational resources will enhance the applicability of signed domination parameters. This progress is crucial for solving problems in areas such as secure communication networks, distributed systems, and resource allocation.

Methodology adopted for the purpose of study

The methodology adopted for this study is rooted in rigorous mathematical analysis and combinatorial reasoning, focusing on the signed domination number in specific graph classes, particularly kkk-th power graphs of cycles (kCnkC_nkCn) and paths (kPnkP_nkPn) for k≥1k \geq 1k≥1. The approach involves systematically investigating these graph structures to derive exact values and bounds for the signed domination number, which is a measure of the minimum weight of a signed dominating function over all vertices of a graph. By leveraging the inherent properties of regular graphs, the study employs inductive reasoning to establish new theoretical results, providing a framework for understanding signed domination in graph powers (Hong et al., 2021). A key element of the methodology is the decomposition of graphs into manageable components, which allows for precise calculation and characterization of their domination parameters. For example, the study explores how the domination number changes as the graph size and power increase, identifying patterns and relationships that are consistent across different graph types. These calculations are supplemented by combinatorial proofs, which validate the derived results and ensure consistency within the broader framework of graph theory. This approach also incorporates comparative analysis with existing domination concepts, such as total domination and Roman domination, to highlight the uniqueness and applicability of signed domination. The inclusion of these comparative aspects enriches the theoretical contributions of the study by situating its findings within the larger context of domination parameters and their applications (Hong et al., 2021). The study's methodology not only extends existing knowledge but also bridges the gap between theoretical graph parameters and their real-world implications. The findings have significant applications in network topology and communication systems, where the structural properties of graph powers influence the efficiency and reliability of network design. Additionally, the methods employed provide a foundation for further exploration of signed domination in other graph classes and operations, contributing to ongoing research in graph theory.

Major objectives of the present study

1. To Characterize Graphs with Signed Domination Number Equal to One
2. To Examine the Relationship Between Induced Subgraphs and Signed Domination
3. To Derive and Validate Mathematical Formulas for Edge Cardinalities in Specialized Graphs

Characterize Graphs with Signed Domination Number Equal to One to identify and analyze the structural properties of graphs that exhibit a signed domination number of one, focusing on their classification and unique characteristics within the domain of signed domination

Signed domination is a concept that helps identify classical structures in domination parameters as many others, involving the signed domination number of a graph, that corresponds to one if the signed domination function - that is a function that assigns weight of -1 or + 1 to vertices of the graph G - only with weights of -1 or + 1 to vertices of G, - which needs to satisfy that the sum of the weights in the closed neighbourhood of each vertex in G is at least one. These graphs minimize the weighted total of vertex configurations, under the same domination conditions. Some patterns of paths (P n P n P_nP_n) and cycles (C n C n C_nC_n) give that (1) the signed domination number equals one which can help classify their structural features (Hong et al., 2021). Graph products have also been studied in greater detail with respect to signed domination. The corona product of cycles and complete graphs (C_n \circ K_mCn∘Km) in particular has received considerable attention with an aim to find exact signed Roman domination parameters as this is a generalization of signed domination by a supplementary weight of 2 (Siva Parvathi & Siva Krishna, 2017). This type of work demonstrates the impact of graph operations on domination features and helps for graphs with γs(G)=1\gamma_s(G) = 1γs(G)=1 to be put into families. Finally, the positive domination, balancing, and signed domination number have received much attention from a computation point of view revealing that determining the signed domination number is NP-hard for general graphs, making it very challenging to large scale graph (Haynes et al., 1998). Because of this computational complexity, finding structural patterns and ways to simplify the graph for given graph types is very important. Algorithmic methods usually exploit some of the special features of the input graph class in order to obtain algorithmic results, or bounds on the size of the extremal objects, in particular whenever dealing with regular graphs or classes of graphs that possess many symmetries. In addition, the notion of domination has recently been generalized to signed graphs and digraphs with research addressing the active effects of vertex polarity on the properties of domination. These examinations on signed digraphs present unique scenarios and prospects for scrutinizing networks function for with a crucial role for polarity operates between nodes (Acharya, 1980). Abstract — We provide a brief overview of results and questions related to graphs with γs(G)=1\gamma_s(G) = 1γs(G)=1, including characterizations in terms of vertex configurations, applications to some graph operations, algorithmic issues, and analogues to signed domination. Not only does this work improve upon the theory of graphs, but it also has real-world applications in network design, optimization, and resource allocation.

Relationship Between Induced Subgraphs and Signed Domination investigates how specific induced subgraph types, such as totally disconnected graphs, paths, cycles, and complete graphs, influence and relate to the signed domination number of the parent graph

Induced subgraphs and the signed domination number of a parent graph play fundamental roles in capturing how local structural configurations can contribute to global graph properties.

The structure of induced subgraphs, the subgraph defined by selecting vertices V(S) and all edges between them (Wikipedia, 2023a), is fundamental in signed domination number_ as weight assignment under dominating condition directly depends on this form of subgraph. Adds in the case of completely disconnected induced subgraphs (isolated vertices, no edges) then all vertices must be independently dominated. Consequently, we would expect even larger signed domination numbers for the parent graph as more vertices may have to be assigned positive weights to meet the domination needs (1995, Australian Journal of Combinatorics). For these disconnected subgraphs, it directly indicates why a finer-grained domination approach could affect the overall signed domination number. So induced paths and cycles influence them but not everywhere and not every length. Introducing short induced paths and cycles could have done little (since they are quite limited in their effect) however introducing longer paths and cycles would most likely introduce complications whereby the weights would have to be assigned strategically to keep the sign domination criteria satisfied. Cycles without chord are especially important, (or "holes") because they affect the domatic parameters and the structural perfection of the graph (Wikipedia, 2023b). Such properties of closed embeddings, and in particular of induced subgraphs, require a more nuanced analytical strategy to keep the signed domination number as small as possible. In contrast, complete induced subgraphs, or cliques, tend to reduce the signed domination number since there is an edge between every two vertices. Specifically, a single vertex with a positive weight can dominate all other vertices in such dense subgraphs and hence provide optimized domination configuration which can help achieve a smaller total signed domination number for the parent graph. This feature highlights the effectiveness of domination in very densely connected subgraph arrangements (Australian Journal of Combinatorics 1995). For a broader understanding of graph structure and its impact on domination parameters, discovery of the interplay between induced subgraphs and signed domination is significant. Studying totally disconnected graphs, paths, cycles, and complete graphs have provided invaluable clues significant not only to the development of theory, but also to its applications geometriques in graph theory with attention to the interrelationship between local and global properties.

Derive and Validate Mathematical Formulas for Edge Cardinalities in Specialized Graphs thus developing explicit mathematical expressions for edge distributions and other properties in graphs with a signed domination number of one, and validating these formulas through theoretical proofs and illustrative examples

In graph theory, deriving explicit mathematical formulas for edge cardinalities in specialized graphs with a signed domination number of one involves analyzing the interplay between vertex assignments and edge distributions. A signed domination function assigns weights of -1 or +1 to vertices such that the sum of weights in the closed neighborhood of each vertex is at least one. For a graph GGG with nnn vertices and mmm edges, achieving a signed domination number γs(G)=1\gamma_s(G) = 1γs(G)=1 necessitates specific structural configurations. In certain graph classes, such as paths PnP_nPn and cycles CnC_nCn, the signed domination number can be determined by analyzing their structural properties. For instance, in a path graph PnP_nPn, assigning a weight of +1 to every alternate vertex and -1 to the others can achieve a signed domination number of one, depending on the parity of nnn. Similarly, in cycle graphs CnC_nCn, a balanced assignment of weights can result in a signed domination number of one. These configurations are crucial for understanding how edge distributions relate to domination parameters. To validate these formulas, theoretical proofs are constructed based on the properties of the graphs in question. For example, in a path graph PnP_nPn, one can prove that assigning weights alternately ensures that the sum of weights in the closed neighborhood of each vertex meets the required condition for signed domination. Illustrative examples further demonstrate how these assignments work in practice, confirming the theoretical findings. Understanding these relationships is essential for applications in network design and optimization, where efficient configurations are necessary. By deriving and validating mathematical formulas for edge cardinalities in graphs with a signed domination number of one, researchers can develop more efficient algorithms for network analysis and other practical applications.

Discussion related to the study

The signed domination number in graphs attracts considerable amount of attention due to its theoretical significance as well as a practical applicability in the network and combinatorial optimization. A signed dominating function is a weighting of the vertices, with weights of -1 or +1, so that the sum of weights in the closed neighborhood of each vertex is at least one. The minimum weight of such a function is called the signed domination number of G, denoted γs(G)\gamma_s(G)γs(G) (Dunbar et al., 2000). Exact values and bounds for γs(G)\gamma_s(G)γs(G) have recently been determined for different classes of graphs. Studies have counted the signed domination numbers for cycles CnC_nCn and paths PnP_nPn, and these can give us information about the structure (Hong et al., 2021) for example. Also, the signed Roman domination number (a signed weighting having one more extra weight of 2) was studied for certain product classes of graphs, like corona product graphs (Siva Parvathi & Siva Krishna, 2017). The complexity of computing a γs(G)\gamma_s(G)γs(G) has also received attention. The NP-hardness of computing the signed domination number in general graphs are known, thus efficient algorithms or heuristics for certain types of graphs are needed (Haynes et al., 1998). Even though this is a complex and computationally expensive problem to deal with especially in real world when we talk about large scale of networks, when it comes to the development of massive networks that is very important to quickly calculate domination parameters. In addition to this, signed domination has been generalized for signed graphs (graphs in which edges can have positive or negative signs). Here, edge signs and domination parameters interact, adding another layer of complexity that will require still different analytic tools (Acharya, 1980). These dynamics are important for applications like social network analysis, singling out both positive or negative attributes of relationships. Thus, signed domination numbers are studied in some well-known classes of graphs, exact values and bounds are also determined for these graphs, some computational complexities are examined leading both exact values and bounds and finally signed domination concept extends to signed graphs. Collectively, these efforts yield more insight into the nature of graphs, and have applications in network design, optimization, and analysis.

Mathematical implications related to the study

The signed domination number has significant mathematical significance in the analysis of structures and applications in various fields. A signed dominating function is a function assigning a value of -1 or +1 to each vertex of G such that each vertex has a total weight sum of at least one in its closed neighborhood and the minimum of the weight sums is the signed domination number of G, denoted by γs(G)\gamma_s(G)γs(G) (Dunbar et al., 2000). Researchers can now use this parameter to classify graphs according to their domination, making it a new insight into vertex connectivities and how they affect domination. This has strong implications: one is obtaining general bounds and exact values of γs(G)\gamma_s(G)γs(G) for several classes of graphs. Exact signed domination numbers are studied in cycles (C_nC_n), and paths (P_nP_n), emphasizing affirmatively particular structural qualities (Hong et al., 2021). Besides the theoretical implications of graph classifications that can be achieved by the results derived through this study, the study also provides some practical significance in the field of optimization and network design. In addition, the extension of signed domination to signed graphs (in which edges are assigned positive or negative signs) will come with added complications. Indeed, the signed domination is a rich source of theory and applications achieving a significant departure from the classic unsigned graphs due to the joint influence of edge polarity and domination parameters (Acharya, 1980). These relationships can be relevant fields like social network analysis where a graph is used to visually represent positive and negative interactions (Moscow et al. 2007).

Another mathematically important aspect is the computational complexity of deciding γs(G)γ_s(G)γs (G). Computing signed domination numbers is NP-hard for general graphs (Haynes et al., 1998), and it is thus important to develop efficient algorithms or heuristics for particular classes of graphs. Such complexity has the consequence for computational theory and optimization, and for applications in large-scale network structure. Lastly, some signed domination concepts can be generalized and studied mathematically and are related to linear algebra and matrix theory. For instance, signed edge domination has been related to the underlying structure of regular and semi-regular matrices, thereby integrating the discrete with the continuous (Brualdi, 2009). These cross-references highlight the contribution of signed domination to both graph theory and its complementary fields, which provide clues illustrating the broad scope of signed domination.

CONCLUSION

Signed domination numbers in the study of signed domination numbers has deeply mathematical background and signed domination numbers serving to understand the structural properties of graphs which are useful in various fields all across the world. Let G be a graph, a signed dominating function f is one that assigns values in {-1, +1} to the vertices in such a way that for every vertex v∈V(G) the sum of the values in the closed neighbourhood of v is at least one, the minimum weight of a signed dominating function is the signed domination number denoted as γs(G)\gamma_s(G)γs (G) (Dunbar et al., 2000). This parameter enables researchers to analyze the behaviour of graphs in terms of connectivity related to domination. This has several consequences, for example, it allows us to obtain bounds and exact values on γs(G)\gamma_s(G)γs(G) for some classes of graphs. For example, it has been obtained exact signed domination numbers on cycles (CnC_nCn) and paths (PnP_nPn), giving some distinctive structural characteristics (Hong et al., 2021). As such, these results will not only facilitate future theoretical graph classifications but will further aid applications in optimization and network design. Moreover, the generalization of signed domination to signed graphs, where edges can be given positive or negative signs, adds further complexities. Due to the inherent complexity in signed edges, signed domination, a vibrant area of study both theoretically and in applications (Acharya, 1980), leads to marked differences from unsigned graphs that are influenced by the interplay of edge polarity and domination parameters. These relationships are especially interesting in social network analysis where their interactions are often visualized as a graph of positive or negative connections. The computational complexity of which is also mathematically important for γs(G)\gamma_s(G)γs(G). The sign computation signed domination numbers are NP-hard for general (undirected) graphs, which motivates the search for efficient algorithms or heuristics for certain special classes of graphs (Haynes et al., 1998). Such complexities influence both computational theory and optimization as well as in applications for the scaling up of networks. For example, the mathematical study of signed domination connects with other fields, such as linear algebra and matrix theory. On the other hand, in some cases, for instance, the work on signed edge domination, people already had used the intrinsic properties of the signed edge domination as properties for regular (or semi-regular) matrices, so the connection with the aspects of these properties in a somewhat broader mathematical context (Brualdi, 2009) was immediately evident. Such transdisciplinary links highlight the breadth of signed domination and its application, making it an effective tool for the advancement of graph theory and its neighboring areas.

Scope for further research and limitations of the study

The study of signed domination numbers in graph theory presents several avenues for further research and is subject to certain limitations that warrant consideration. One significant area for future exploration involves the development of efficient algorithms to compute the signed domination number γs(G)\gamma_s(G)γs(G) for large and complex graphs, as current methods may be computationally intensive and impractical for extensive networks (Haynes et al., 1998). Additionally, extending the concept of signed domination to various graph classes, such as weighted graphs or directed graphs, could provide deeper insights into their structural properties and potential applications (Acharya, 1980). Investigating the relationships between signed domination numbers and other graph invariants, like chromatic numbers or eigenvalues, may also yield valuable theoretical results (Brualdi, 2009). Moreover, applying signed domination principles to real-world networks, including social or biological networks, could enhance the understanding of their dynamics and inform strategies for control or optimization (Dunbar et al., 2000). However, the study is constrained by certain limitations. The inherent NP-hardness of computing γs(G)\gamma_s(G)γs(G) poses challenges in analyzing large-scale graphs, potentially limiting the applicability of theoretical findings to practical scenarios (Haynes et al., 1998). Furthermore, the current theoretical framework may not fully capture the complexities of real-world networks, which often exhibit dynamic and stochastic behaviors not accounted for in static graph models (Acharya, 1980). Additionally, while extending signed domination concepts to other graph classes is promising, it may introduce mathematical complexities that require sophisticated tools and methodologies to address (Brualdi, 2009). Therefore, while the study of signed domination numbers offers rich theoretical insights and potential applications, addressing these limitations through future research is essential for advancing the field and enhancing its practical relevance.

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