

A Study on Properties and Applications of Elzaki Transformation

GOVIND RAJ NAUNYAL¹, UPDESH KUMAR², DINESH VERMA³

^{1, 2} Department of Mathematics, KGK (PG) College Moradabad

Abstract- These are considerably long in comparison with their lateral dimensions and hence buckle when the axial load approaches a certain critical value known as critical buckling load. In this paper, we present Elzaki transformation means for discussing the Euler's theory of very long columns with low critical buckling loads to obtain the Euler's formula for critical or buckling load. It is a powerful mathematical means which is generally applied in different areas of science, engineering and technology for solving ordinary or partial differential equations without finding their general solutions.

Indexed Terms- Euler's theory, Elzaki transformation, Long Columns, Critical buckling load.

I. INTRODUCTION

Elzaki Transformation applied in solving boundary value problems in most of the science and engineering disciplines [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. It also comes out to be very effective tool to analyze differential equations with delta function [12,13, 14,15,16,17]. The differential equations are generally solved by adopting Laplace transform method or convolution method of residue theorem method [18,19, 20,21,22,23, 24]. In this paper, we present a new technique called Elzaki transform to analyze differential equations with delta function.

BASIC DEFINITIONS

2.1 Elzaki Transform

If the function $f(y)$, $y \geq 0$ is having an exponential order and is a piecewise continuous function on any interval, then the Elzaki transform of $f(y)$ is given by

$$E\{f(y)\} = \bar{f}(p) = p \int_0^{\infty} e^{-\frac{y}{p}} f(y) dy.$$

The Elzaki Transform [1, 2, 3] of some of the functions are given by

- $E\{y^n\} = n! p^{n+2}$, where $n = 0, 1, 2, \dots$
- $E\{e^{ay}\} = \frac{p^2}{1-ap}$,
- $E\{\sin ay\} = \frac{ap^3}{1+a^2p^2}$,
- $E\{\cos ay\} = \frac{ap^2}{1+a^2p^2}$,
- $E\{\sinh ay\} = \frac{ap^3}{1-a^2p^2}$,
- $E\{\cosh ay\} = \frac{ap^2}{1-a^2p^2}$.

2.2 Inverse Elzaki Transform

The Inverse Elzaki Transform of some of the functions are given by

- $E^{-1}\{p^n\} = \frac{y^{n-2}}{(n-2)!}$, $n = 2, 3, 4 \dots$
- $E^{-1}\{\frac{p^2}{1-ap}\} = e^{ay}$
- $E^{-1}\{\frac{p^3}{1+a^2p^2}\} = \frac{1}{a} \sin ay$
- $E^{-1}\{\frac{p^2}{1+a^2p^2}\} = \frac{1}{a} \cos ay$
- $E^{-1}\{\frac{p^3}{1-a^2p^2}\} = \frac{1}{a} \sinh ay$
- $E^{-1}\{\frac{p^2}{1-a^2p^2}\} = \frac{1}{a} \cosh ay$

2.3 Elzaki Transform of Derivatives

The Elzaki Transform [1, 2, 3] of some of the Derivatives of $f(y)$ are given by

- $E\{f'(y)\} = \frac{1}{p} E\{f(y)\} - p f(0)$

$$\text{or } E\{f'(y)\} = \frac{1}{p} \bar{f}(p) - p f(0),$$

- $E\{h''(y)\} = \frac{1}{p^2} \bar{h}(p) - h(0) - p h'(0)$,
and so on.

METHODOLOGY

Let a very long vertical column AB of length ‘a’ and having uniform area of cross-section, where A is the upper end of column and B is its lower end. Let ‘y’ be the lateral deflection of a section of the column at a distance ‘x’ from the lower end B, I be the moment of inertia of the section, ‘Y’ is the Young’s modulus of elasticity of the column and ‘P’ be the critical buckling load. Now we will discuss four different cases:

Case-I: When both ends A and B of the column are pinned or hinged

In this case, the bending moment [1] at the section is given by

$$\ddot{y}(x) + k^2 y(x) = 0, \text{ where } k = \sqrt{\frac{P}{YI}} \quad (4)$$

Taking Elzaki Transform of equation (4), we get

$$E[\ddot{y}(x)] + k^2 E[y(x)] = 0$$

This equation gives

$$\frac{1}{p^2} \bar{y}(p) - y(0) - p \dot{y}(0) + k^2 \bar{y}(p) = 0 \quad (5)$$

Applying boundary condition: $y(0) = 0$, equation (5) becomes,

$$\begin{aligned} \frac{1}{p^2} \bar{y}(p) - p \dot{y}(0) + k^2 \bar{y}(p) &= 0 \text{ or} \\ \frac{1}{p^2} \bar{y}(p) + k^2 \bar{y}(p) &= p \dot{y}(0) \end{aligned} \quad (6)$$

In this equation, $\dot{y}(0)$ is some constant. Substitute $\dot{y}(0) = A$, equation (6) becomes

$$\frac{1}{p^2} \bar{y}(p) + k^2 \bar{y}(p) = Ap \quad (7)$$

$$\text{Or } \bar{y}(p) = \frac{Ap^3}{(1+p^2 k^2)}$$

Taking inverse Elzaki transforms [3] of equation (7), we get

$$y(x) = \frac{A}{k} \sin(kx) \quad (8)$$

Applying boundary condition: $y(a) = 0$, equation (8) gives $\frac{A}{k} \sin(k a) = 0$

Since A cannot be equal to zero because for $A = 0$, $y = 0$. This means that column will not bend at all, which is not possible.

Therefore, $\sin(k a) = 0$

Or $ka = n \pi$, where n is an integer greater than equal to zero.

$$\text{Or } k = \frac{n\pi}{a} \quad (9)$$

The least practical value of n is 1, therefore considering $n = 1$, we have

$$k = \frac{\pi}{a}$$

$$\text{Or } \sqrt{\frac{P}{YI}} = \frac{\pi}{a}$$

$$\text{Or } P = \frac{\pi^2 YI}{a^2}$$

This equation provides the Euler’s formula for critical buckling load for very long column which is pinned at its both the ends.

Case-II: When lower end B of the column is fixed and the other end A is free

In this case, the bending moment [1] at the section is given by

$$\ddot{y}(x) + k^2 [d - y(x)] = 0, \text{ where ‘d’ is the deflection at the free end A due to critical buckling load.}$$

Taking Elzaki Transform of equation (4), we get

$$E[\ddot{y}(x)] + k^2 E[y(x) - d] = 0$$

This equation gives

$$\frac{1}{p^2} \bar{y}(p) - y(0) - p \dot{y}(0) + k^2 \bar{y}(p) = k^2 p^2 d \dots (5)$$

Applying boundary conditions: $y(0) = 0$ and $\dot{y}(0) = 0$ as the slope at $x = 0$ is zero, equation (5) becomes,

$$\frac{1}{p^2} \bar{y}(p) + k^2 \bar{y}(p) = k^2 p^2 d$$

$$\text{Or } \bar{y}(p) = \frac{k^2 p^4 d}{(1+p^2 k^2)} \quad (6)$$

$$\text{Or } \bar{y}(p) = \frac{k^2 p^4 d}{(1+p^2 k^2)} \quad (7)$$

$$\text{Or } \bar{y}(p) = \frac{d}{p} - \frac{dp}{(p^2 + k^2)}$$

Taking inverse Laplace transforms [4] of equation (7), we get

$$y(x) = \frac{d}{k^2} - \frac{d}{k^2} \cos(kx)$$

(8)

Applying boundary condition: $y(a) = d$, equation (8) gives

$$d = d - d \cos(k a)$$

$$\text{Or } \cos(k a) = 0$$

or $ka = \frac{(2n-1)\pi}{2}$, where n is an integer greater than equal to zero.

$$\text{Or } k = \frac{(2n-1)\pi}{2a}$$

The least practical value of n is 1, therefore considering n = 1, we have

$$k = \frac{\pi}{2a}$$

$$\text{Or } \sqrt{\frac{P}{YI}} = \frac{\pi}{2a}$$

$$\text{Or } P = \frac{\pi^2 YI}{4a^2}$$

This equation provides the Euler's formula for critical buckling load for very long column whose lower end is fixed and upper end is free.

Case-III: When both the ends A and B of the column are fixed

In this case, the bending moment [1] at the section is given by

$$\ddot{y}(x) + k^2 y(x) = M, \text{ Where } M = \frac{M_0}{EI}, M_0 \text{ is the restraint moment at each end.}$$

Taking Elzaki Transform of equation (4), we get

$$E[\ddot{y}(x)] + k^2 E[y(x)] = M$$

This equation gives

$$\frac{1}{p^2} \bar{y}(p) - y(0) - p\dot{y}(0) + k^2 \bar{y}(p) = M E\{1\} \quad (5)$$

Applying boundary conditions: $y(0) = 0$ and $\dot{y}(0) = 0$ as the slope at $x = 0$ is zero, equation (5) becomes,

$$\frac{1}{p^2} \bar{y}(p) + k^2 \bar{y}(p) = Mp^2$$

$$\text{Or } \bar{y}(p) = \frac{Mp^4}{(1+p^2 k^2)} \quad (7)$$

$$\text{Or } \bar{y}(p) = Mp^2 - \frac{Mp^2}{(1+p^2 k^2)}$$

Taking inverse Laplace transforms [5] of equation (7), we get

$$y(x) = \frac{M}{k^2} - \frac{M}{k^2} \cos(kx) \quad (8)$$

Applying boundary condition: $y(a) = 0$, equation (8) gives $\frac{M}{k^2} - \frac{M}{k^2} \cos(k a) = 0$

$$\text{Therefore, } \cos(k a) = 1$$

Or $ka = 2n\pi$, where n is an integer greater than equal to zero.

$$\text{Or } k = \frac{2n\pi}{a} \quad (9)$$

The least practical value of n is 1, therefore considering n = 1, we have

$$k = \frac{2\pi}{a}$$

$$\text{Or } \sqrt{\frac{P}{YI}} = \frac{2\pi}{a}$$

$$\text{Or } P = \frac{4\pi^2 YI}{a^2} \quad (10)$$

This equation provides the Euler's formula for critical buckling load for very long column whose both ends are fixed.

Case-IV: When lower end B of the column is fixed and the upper end A is hinged or pinned

In this case, the bending moment [1] at the section is given by

$$\ddot{y}(x) + k^2 y(x) = H(a - x), \text{ where } H = \frac{H_0}{EI}, H_0 \text{ is horizontal force at the fixed end } B.$$

Taking Elzaki Transform of equation (4), we get

$$E[\ddot{y}(x)] + k^2 E[y(x)] = HE(a - x)$$

This equation gives

$$\frac{1}{p^2} \bar{y}(p) - y(0) - p\dot{y}(0) + k^2 \bar{y}(p) = HE[(a - x)] \quad (5)$$

Applying boundary conditions: $y(0) = 0$ and $\dot{y}(0) = 0$ as the slope at $x = 0$ is zero, equation (5) becomes,

$$\frac{1}{p^2} \bar{y}(p) + k^2 \bar{y}(p) = H \left[\left(\frac{a}{p} - \frac{1}{p^2} \right) \right] \quad (6)$$

$$\text{Or } \bar{y}(p) = H \left[\frac{ap}{(1+p^2 k^2)} - \frac{1}{(1+p^2 k^2)} \right] \quad (7)$$

$$\text{Or } \bar{y}(p) = H \left[\frac{ap}{k^2} - \frac{ap^2}{k^2(1+p^2 k^2)} - \frac{p^3}{k^2} + \frac{ap^3}{k^3(1+p^2 k^2)} \right]$$

Taking inverse Laplace transforms [6] of equation (7), we get

$$y(x) = H \left[\frac{a}{k^2} - \frac{a}{k^2} \cos(kx) - \frac{x}{k^2} + \frac{\sin kx}{k^3} \right] \quad (8)$$

Applying boundary condition: $y(a) = 0$, equation (8) gives

$$H \left[\frac{a}{k^2} - \frac{a}{k^2} \cos(ka) - \frac{a}{k^2} + \frac{\sin ka}{k^3} \right] = 0$$

$$\text{Or } \left[-\frac{a}{k^2} \cos(ka) + \frac{\sin ka}{k^3} \right] = 0$$

$$\text{Or } \frac{a}{k^2} \cos(ka) = \frac{\sin ka}{k^3}$$

$$\text{Or } \tan(ka) = ka$$

On expanding $\tan ka$ upto 5th power of ka and solving we get

$$\text{Or } ka = 4.5 \text{ radians}$$

$$\text{Or } \sqrt{\frac{P}{YI}} a = 4.5 \text{ radians}$$

$$\text{Or } P = \frac{20.25Yl}{a^2}$$

$$\text{Or } P = \frac{2\pi^2Yl}{a^2}$$

This equation provides the Euler's formula for critical buckling load for very long column whose lower end is fixed and upper end is pinned.

CONCLUSION

This paper discussed the Euler's theory of very long columns with low buckling axial loads by means of Elzaki transformation tool. An attempt has made to exemplify the Elzaki transformation method for discussing the Euler's theory of very long columns with low buckling axial loads for obtaining the Euler's formula of critical buckling load. In all the cases discussed, we found that the critical buckling load for very long columns which are subjected to axial loads, is inversely proportional to the square of the length of the column.

REFERENCES

- [1] Dinesh Verma, Elzaki –Laplace Transform of some significant Functions, Academia Arena, Volume-12, Issue-4, April 2020..
- [2] Dinesh Verma, Aftab Alam, Analysis of Simultaneous Differential Equations By Elzaki Transform Approach, Science, Technology And Development Volume Ix Issue I January 2020.
- [3] Sunil Shrivastava, Introduction of Laplace Transform and Elzaki Transform with Application (Electrical Circuits),International Research Journal of Engineering and Technology (IRJET), volume 05 Issue 02, Feb-2018.
- [4] Tarig M. Elzaki, Salih M. Elzaki and Elsayed Elnour, On the new integral transform Elzaki transform fundamental properties investigations and applications, global journal of mathematical sciences: Theory and Practical, volume 4, number 1(2012).
- [5] Dinesh Verma and Rahul Gupta, Delta Potential Response of Electric Network Circuit, Iconic Research and Engineering Journal (IRE) Volume-3, Issue-8, February 2020.
- [6] Dinesh Verma, Elzaki Transform of some significant Infinite Power Series, International Journal of Advance Research and Innovative Ideas in Education (IJARIIE) Volume-6, Issue-1, February 2020.
- [7] Dinesh Verma and Rahul Gupta, Delta Potential Response of Electric Network Circuit, *Iconic Research and Engineering Journal (IRE)* Volume-3, Issue-8, February 2020.
- [8] Rohit Gupta, Dinesh Verma and Amit Pal Singh, Double Laplace Transform Approach to the Electric Transmission Line with Trivial Leakages through electrical insulation to the Ground, Compliance Engineering Journal Volume-10, Issue-12, December 2019.
- [9] Rohit Gupta, Rahul Gupta and Dinesh Verma, Laplace Transform Approach for the Heat Dissipation from an Infinite Fin Surface, Global Journal of Engineering Science and Researches (GJESR), Volume-06, Issue-2(February 2019).
- [10] Dinesh Verma, Elzaki transform approach to differential equation with Legendre polynomial, international research journal of modernization in engineering technology and science (IRJMETS), Volume-2, Issue-3, March -2020.
- [11] Dinesh Verma, Rohit Gupta, Analysis boundary value problems in physical science via elzaki Transform, ASIO journal of chemistry, physics, Mathematics and applied sciences (ASIO JCPMAS), VOLUME-4, Issue-1,2020.
- [12] Dinesh Verma, Aftab Alam, Dinesh Verma – Laplace Transform of some momentous Function,Advances and Applications in mathematical sciences,Volume 20,Issue 7, may 2021.
- [13] Dinesh verma, Rahul gupta, rohit Gupta, determining rate of heat convected from a uniform infinite fin using gupta transform, Roots international journal of multidisplenary researchers,volume 7, issue 3, February 2021.
- [14] Dinesh Verma Analytical Solution of Differential Equations by Dinesh Verma Transforms (DVT), ASIO Journal of Chemistry, Physics, Mathematics & Applied Sciences (ASIO-JCPMAS), Volume -4, Issue-1, 2020, PP:24-27.
- [15] Dinesh Verma, Empirical Study of Higher Order Differential Equations with Variable Coefficient

- by Dinesh Verma Transformation (DVT), ASIO Journal of Engineering & Technological Perspective Research (ASIO-JETPR), Volume - 5, Issue-1, 2020, pp:04-07.
- [16] Dinesh Verma, Amit Pal Singh and Sanjay Kumar Verma, Scrutinize of Growth and Decay Problems by Dinesh Verma Tranform (DVT), Iconic Research and Engineering Journals (*IRE Journals*), Volume-3, Issue-12, June 2020; pp: 148-153.
- [17] Dinesh Verma, Elzaki Transform Approach to Differential Equations, *Academia Arena*, Volume-12, Issue-7, 2020, pp: 01-03.
- [18] Dinesh Verma and Rohit Gupta, Analyzing Boundary Value Problems in Physical Sciences via Elzaki Transform, by in ASIO Journal of Chemistry, Physics, Mathematics & Applied Sciences (ASIO-JCPMAS), Volume -4, Issue-1, 2020, ISSN: 2455-7064, PP:17-20.
- [19] Dinesh Verma, Elzaki Transform Approach to Differential Equatons with Leguerre Polynomial, *International Research Journal of Modernization in Engineering Technology and Science (IRJMETs)*, Volume-2, Issue-3, March 2020, pp: 244-248:
- [20] Theory of Structures by S. Ramamrutham and R. Narayan, Danpat Rai Publishing Company.
- [21] Advanced Engineering Mathematics by Erwin Kreysig 10th edition, 2014.
- [22] Rohit Gupta, Rahul Gupta, Dinesh Verma, Eigen Energy Values and Eigen Functions of a Particle in an Infinite Square Well Potential by Laplace Transforms, *International Journal of Innovative Technology and Exploring Engineering (IJITEE)*, Volume-8 Issue-3, January 2019, pp. 6-9.
- [23] Rahul Gupta, Rohit Gupta, Dinesh Verma, "Application of Convolution Method to the Impulsive Response of A Lightly Damped Harmonic Oscillator", *International Journal of Scientific Research in Physics and Applied Sciences*, Vol.7, Issue.3, pp.173-175, June (2019).