Logistic Brownian Motion with Jumps

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Abstract- Black and Scholes [1973] approach to option price estimations and option trading brought about a great breakthrough in financial mathematics. Since Black and Scholes [1973], the standard model in financial mathematics has been the Geometric Brownian motion. In this model it is assumed that the asset's log return has a normal distribution with volatility and drift terms. The model has proved to have very attractive features. However, from empirical study, geometrical Brownian motion cannot accurately reflect all behaviors of the stock quotation. The model has some limitations in price prediction, especially when used to model the price over short period of time. The study involves derivation of logistic Brownian motion with jump diffusion for a better study of the behavior of the underlying asset.

I. INTRODUCTION

The aim of modifying the Black-Scholes model is to incorporate two major empirical features of financial markets:

1.1 The leptokurtic distribution features
The market trends indicate that the return distribution has a higher peak and two heavier tails unlike that of the normal distribution. The leptokurtic distribution means the kurtosis of the distribution is large.

1.2 The volatility smile
In Black-Scholes formula, the call/put options are monotone increasing functions of volatility. The implied volatility $\sigma(T, K)$is a parameter associated with a particular strike price K and a particular maturity time T such that if the volatility parameter is used in the Black-Scholes formula for European call/put options, then we shall obtain a price that exactly matches the market price of a particular call/put options. More precisely $\sigma(T, K)$is the inverse function of the market option price in terms of volatility. When implied volatility is plotted against different strike prices a convex curve formed resembles a 'smile'. This is contrary to the famous Black-Scholes model as it assumes that the volatility is constant. In reality the implied volatility is not constant. If it was so, then the observed implied volatility curve should look flat (Derman and Kani[1994]).

A jump diffusion model consists of two parts; a diffusion component modeled by a Brownian motion describing the instantaneous part of unanticipated return due to normal price vibrations and a jump component modeled by a Poisson process describing the part due to the abnormal price vibrations. The asset price jumps are assumed to be independently and identically distributed. Generally, the jump diffusion model is of the form;

$$dS(t) = \mu S(t)dt + \sigma S(t)dZ(t) + \rho S(t)dq(t) , \quad (1)$$

Where, $S(t)$ is the asset price at a time t, $dZ(t)$ is the Brownian motion process, $q(t)$ is a Poisson process with an intensity of λ, ρ is an impulse function which causes a jump from $S(t)$ to $S(1+\rho)$.

II. PRELIMINARIES

In this section we discuss two models which were modified from the famous Black-Scholes models.

2.1 Merton Jump diffusion model

Merton [1975] was also involved in the process of developing the Black-Scholes model and came up with Merton jump model as a better estimation of option prices in a precise way. The Merton model has the same assumptions as those of BlackScholes except how the asset price is modeled. This model where the asset price has jumps superimposed upon a geometric Brownian motion is given by

$$dS(t) = S(t)(\mu - \lambda k)dt + \sigma S(t)dZ(t) + S(t)(q - 1)dq(t) , \quad (2)$$

where μ is expected return from the asset, λ is the rate at which jumps happen and $k = \ k = \varepsilon(q-1)$ is the average jump size measured as a proportional increase in the asset price. $q-1$ is the random variable percentage change in the asset price if the Poisson event occurs, ε is the expectation operator over the random variable q, $dZ(t)$ is the change in Brownian motion process and $dq(t)$ is the independent Poisson process generating the jumps. μdt is adjusted by λkdt in the drift term to make the jump part unpredictable innovation Merton [1975]. Merton gives an insight on how we can estimate and come up with option pricing model when a mixture of both continuous and jump processes generate underlying stock returns. It demonstrates how total change in stock price is caused by normal price change chat produces a marginal change in price and also abnormal price change which is described by a jump process showing the non-marginal impact due to new information. Merton model shows that stock prices follow log-normal distribution and the probability if a jump occurs or not is modeled by a Poisson process.

The probability of the Poisson process can be described as;

(i) P {the event does not occur in the time interval $(t, t+h)\} = 1 - \lambda\psi h + O(\psi)\}$

(ii) P {the event occurs once in the time interval $(t, t+h)\} = \lambda\psi h + O(\psi)$

(iii) P {the event occurs more than once in the time interval $(t, t+h)\} = O(\psi)$

Therefore, this can be described as;

$\frac{dS(t)}{S(t)} = (\mu - \lambda k)dt + \sigma S(t)dZ(t)$ if the event does not occur and

$\frac{dS(t)}{S(t)} = (\mu - \lambda k)dt + \sigma S(t)dZ(t) + (q-1)dq$ if the event does occur.

Thus if $\lambda = 0$, also $q - 1 = 0$, then the stock price return is equivalent to Black-Scholes and Merton approaches. Solving for (2) gives

$$S(t) = S(0)\exp\left[\left(\mu - \lambda k - \frac{\sigma^2}{2}\right)t + \sigma Z(t)\right]\prod_{i=1}^{i=N(t)} q_i, \quad (3)$$

where $N(t)$ is a poisson process with rate λ, $Z(t)$ is a standard Brownian motion and μ is the drift rate. In this solution q_i is a sequence of independent identically distributed (i.i.d) non-negative random variables. Merton [1975] assumed that $\log(q_i) = Y_i$ is the absolute asset price jump size and log-normally distributed. In other words $\log(q_i) \sim i.id.N($ ~ i.i.d.N (μ, σ^2).

By adding jumps to the Black-Scholes model and choosing the appropriate parameters of the jump process, the log-normal jump models generate volatility smile or skew. When the mean of the jump process is set to be negative, steep short-term skews are easily captured in this framework (Andersen and Andersen [2000]). However, it is difficult to study the first passage times for log-normal jump diffusion Merton model when a jump diffusion crosses boundary level when an overshoot occurs. This makes it impossible to simulate the jump unless the exact distribution of the overshoot is determined.

2.2 Kou double exponential Jump diffusion model

Stephen Kou developed what is referred to us Kou double exponential jump diffusion model. According to this model jumps of stock prices are not log-normally distributed as in the case of Merton, instead jumps follow a double-exponential distribution, (Kou [2002]). The assumptions in this model are the same as those for Merton and Black-Scholes.

Just as in Merton Model, Kou model consists of two parts; The first part is driven by a normal geometric Brownian motion hence its path is continuous. The second part is the jump part with a logarithm of jump size which is double-exponentially distributed and the jump times are determined by a Poisson process. The model is of the form;

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma S(t)dZ(t) + d\left[\sum_{i=1}^{N(t)}(V_i - 1)\right], \quad (4)$$

Where, μ is the expectation value, σ is the volatility and $N(t)$ is a Poisson process with the parameter . $\{V_i\}$ is a series of independently identically distributed nonnegative random variables.

$Y = log(V)$ and has got asymmetric double exponential distribution.

$Z(t)$, $N(t)$ and Y_i are the sources of randomness and are assumed to be independent

Solving equation (4) we obtain;

$$S(t) = S(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z(t)\right]\prod_{i=1}^{i=N(t)} V_i \quad (5)$$

where $S(t)$ is the stock price, $S_0 = S(0)$ is the stock price at time zero, μ is the growth rate, σ is the volatility which depicts the uncertainty of the stock price and $Z(t)$ is the Wienner process and $\{V_i\}$ is series of independently identically distributed nonnegative random variables.

The Kou double exponential jump diffusion model has some advantages in that it yields a closed-form solution for standard European call and put options. In addition it leads to a variety of closed-form solution for path-dependent options such as look back options, barrier options, American options and interest rate derivatives like bond options. The model captures the asymmetric leptokurtic feature and volatility smile. This is able to reproduce the leptokurtic feature of the return distribution and the "volatility smile" as observed in option prices. This makes it fit in stock data better than the normal jump diffusion model, (Kou [2002]).

Another important aspect is that the model has some economical, physical, and psychological interpretations. Many empirical studies have suggested that markets tend to have both overreaction and under reaction to various good and bad news. One may interpret the jump part of the model as the market response to outside news. In simple terms, in the absence of outside news the Kou Double Exponential Jump Diffusion Model asset price simply follows a geometric Brownian motion. Good and bad news arrives according to a Poisson process, and the asset price changes in response according to the jump size distribution. Because the double exponential distribution has both a high peak and heavy tails, it can be used to model both the overreaction (attributed to the heavy tails) and the under reaction (attributed to the high peak) to outside news (Kou and Wang [2004]). Therefore, the double exponential jump diffusion model can be interpreted as an attempt to build a simple model, within the traditional random walk and efficient market framework, to incorporate investors' sentiment, (Kou [2002]).

III. MAIN RESULTS

3.1 Logistic Brownian motion with jump diffusion

The logistic stochastic differential equation and incorporating the jump diffusion process in Geometric Brownian motion is given as;

$$\frac{dS(t)}{S(t)(S^* - S(t))} = (\mu - \lambda k)dt + \sigma dZ + dq \tag{6}$$

Using the Heavyside cover up method on the L.H.S we have;

$$\frac{dS(t)}{S^*S(t)} + \frac{dS(t)}{S^*S^* - (S(t))} = (\mu - \lambda k)dt + \sigma dZ + dq \tag{7}$$

Integrating equation (7) from $t_0$to t gives;

$$\frac{1}{S^*}ln|S(t)| - \frac{1}{S^*}ln|[S^* - S(t)]| = (\mu - \lambda k)t + \sigma Z(t) + q(t) \tag{8}$$

which can re-written as,

$$\ln|\frac{S(t)(S^* - S(0))}{S(0)(S^* - S(t))}| = S^*(\mu - \lambda k)(t - t_0) + S^*\sigma Z(t) + S^*q(t) \tag{9}$$

Solving for $S(t)$ we will finally get

$$S(t) = \frac{S^*S(0)}{S(0) + (S^* - S(0))e^{-((\mu-\lambda k)S^*(t-t_0)+\sigma S^*Z(t)+S^*q(t))}} \tag{10}$$

This price dynamic is referred to as logistic Brownian motion with jump diffusion of stock price $S(t)$, with the initial price $S(0)$, equilibrium price S^*, μ is the expected return from the asset, λ is the rate at which jumps happen and k is the average jump size measured as a proportional increase in asset price and q is the poison process generating jumps. The model (10) is of great use in determining the price of the underlying asset.

CONCLUSION

The model has been modified to represent a non-linear variation of the famous Black-Scholes equation. Non-linear Black-Scholes equation tends to provide a better tool for predicting price changes by taking into account more realistic assumptions than that of the original Black-Scholes. This equation takes care of the transaction costs, illiquid markets, risks from unprotected portfolio and large investor's preferences. These assumptions have a great impact on the stock price, the option price, volatility and the asset's growth rate.

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