An Irreducible and Doubly Even Code of Degree 23 Related to Mathieu Group M_{23}

VINCENT NYONGESA MARANI
Mathematics Department, Kibabii University, Bungoma, Kenya

Abstract- In this paper, we determine an irreducible and doubly even code [23,11,8]. We determine and discuss the properties of designs related to this code

Indexed Terms- Irreducible and Doubly Even Code, Mathieu Group M_{23}.

I. INTRODUCTION

Given a permutation group G acting on a finite set Ω, we find all submodules of the permutation module. The submodules constitute the building blocks for the construction of a lattice of submodules. Let G be the Mathieu group M_{23}. Group G acts on a point to generate the point stabilizer M_{22}. The point stabilizer is a maximal subgroup of degree 23 in G. The group G acts on this maximal subgroup over F2 to form a module of dimension 23. We take the permutation module to be our working module and recursively find all maximal submodules. The recursion stops as soon as we obtain all maximal submodules.

We find that permutation module splits into maximal submodules of dimension 1, 11, 12 and 22. The module breaks down into three completely irreducible parts of length 1, 11, and 11 with multiplicities 1, 1, and 1 respectively. The submodule lattice is as shown in Figure 1

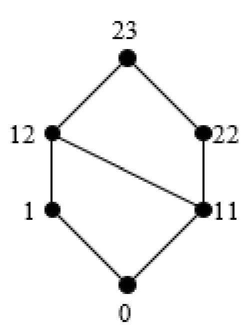

Figure 1: Submodule Lattice of the 23-Dimensional Permutation Module

The submodules of dimensions 1 and 11 are irreducible.

We obtain two non-trivial submodules of dimensions 11 and 12. The binary linear code from this representation is [23,11,8] and its dual [23, 12, 7]. We shall denote the code $C_{23,1}$ and its dual $C^{\perp}_{23,1}$. The weight distribution of these codes is given in Table 1 below.

Table 1: The weight distribution of the codes from a 23-dimensional representation.

name	dim	0	7	8	11	12	15	16	23
$C_{23,1}$	11	1		506		1288		253	
$C^{\perp}_{23,1}$	12	1	253	506	1288	1288	506	253	1

We make some observations about the properties of these codes in Proposition 1.1.

- Proposition 1.1. *Let G be the Mathieu group M_{23} and $C_{23,1}$ a binary code of dimension 11 from a module of degree 23. $C_{23,1}$ is self-orthogonal doubly even projective [23, 11,8] binary code. The dual code $C^{\perp}_{23,1}$ of $C_{23,1}$ is a [23, 12, 7]. Furthermore $C_{23,1}$ is irreducible and Aut(C23,1) ≅ M23.*

- proof

The submodule 12 represents the dimension of binary code $C_{23,1}$. From this submodule we determine the binary linear code [23, 11,8]. The polynomial of this code is $W(x) = 1+506x^8+1288x^{12}+253x^{16}$. From the polynomial we deduce that the weight of codewords are divisible by 4. Therefore $C_{23,1}$ is doubly even. The minimum weight of $C^{\perp}_{23,1}$ code is 7. Hence $C_{23,1}$ is projective. From the lattice structure the submodule 12 is a direct sum of trivial submodules 11 and 1 respectively which implies that $C_{23,1}$ is irreducible. For the structure of the automorphism group, let *G*

$\cong Aut(C_{23,1})$. Composition factors of G are Z1 and M_{23} and the order of G is 10200960. This implies that $G = M_{23}$. Since $M_{23} \subseteq G$, we conclude that $G \cong M_{23}$

II. DESIGNS OF CODEWORDS IN [23,11,8]

We determine designs of codewords t in $C_{23,1}$. Table 2 shows deigns of codewords in $C_{23,1}$ where column one represents the code $C_{23,1}$ of weight m and column two gives the parameters of the 3-designs $D_{wm.}$ In column three we list the number of blocks of D_{wm}, four tests whether or not a design D_{wm} is primitive under the action of Aut(C).

Table 2: Deigns held by the support of codewords in $C_{23,1}$

m	Dw_m	No of Blocks	primitive
8	3-(23, 8, 16)	506	yes
12	3-(23, 12, 160)	1288	yes
16	3-(23, 16, 80)	253	yes

- Remark 2.1. *From the results in table 2 we observe that Aut(C) is primitive on* D_{wm} Maximal subgroups of degree 506, 1288, and 253 are stabilizers in M_{23} and the blocks 506, 1288, and 253 in table 2 represents codewords of weight 8, 12 and 16 respectively.

III. SYMMETRIC 1- DESIGNS

In this section we consider G to be the simple Mathieu group M_{23} and examine symmetric 1-design invariant under G constructed from orbits of the rank - 2 permutation representation of degree 23.

Table 3 shows Symmetric 1-Design where column one represents the 1-design D_k of orbit length k, column two gives the orbit length, column three shows the parameters of the symmetric 1-design D_k and column four gives the automorphism group of the design.

Table 3: Symmetric 1-Design

Design	orbit length	parameters	Automorphism Group
D22	22	1-(23,22,22)	$A_{23:}$ 2

- Proposition 3.1. *Let G be the mathieu simple group* M_{23}, *and* Ω *the primitive G-set of size 23 defined by the action on the cosets of* M_{23}. *Let* $\beta = \{M^g : g \in G\}$ *and* $D_k = (\Omega, \beta)$.

Then the following hold:

i D_k *is a primitive symmetric* $1-(23, |M|, |M|)$ *design.*

ii $Aut(D_k) \cong A_{23:}$ 2

- Proof

i. Since G acts as an automorphism group, primitive on points and on blocks of the design, G ⊆ Aut(D_k).

ii. The order of Aut (D_k) is 25852016738884976640000. The factors of 25852016738884976640000 are 2 and 12926008369442488320000 which corresponds to the composition factors

Z_2 and A_{23} and so Aut (D_k)= $A_{23:}$ 2. This implies that $Aut(D_k)$ u $A_{23:}$ 2

CONCLUSION

Let G be the primitive group of degree 23 of M_{23} and C a linear code admitting G as an automorphism group. Then the following holds:

a) There exists a self-orthogonal irreducible doubly even projective code.

b) There exist a set of Primitive Designs related to M_{23}.

REFERENCES

[1] L. Chikamai, J. Moori, and B. G. Rodrigues (2012), *Linear codes obtained from 2modular representations of some finite simple groups.* thesis D-Space software copyright ' 2002-2013 Duraspace

[2] V.N. Marani (2019), *Some Linear Codes, Graphs and Designs from Mathieu Groups M_{24} and M_{23}.* Ph.D. Thesis, Kibabii University.

[3] B. G. Rodrigues (2003), *Codes of Designs and Graphs from Finite Simple Groups*, Ph.D. thesis, University of Natal, Pietermaritzburg,