Blasiu's Theorem and Its Applications

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Abstract- The basic concepts of fluid motion are expressed Laurent's Theorem, Residue Theorem, Circle theorem are presented-Blasiu's theorem and it's application are also discussed. A source outside a cylinder with and without circulation around the cylinder is described. A doublet outside a cylinder is studied.

Indexed Terms- Moment, Laurent. Residue, Circle Theorem.

I. INTRODUCTION

When a long cylinder is placed with its generators perpendicular to the incident stream of a moving fluid containing hydrodynamic singularities, such as sources, sinks it experience forces tending to produce translation and rotation of the cylinder. These effects are calculated using the following theorem due to Blasius. The cylinder can be of any general section. In practice it can be an aerofoil. We confine attention to the case when the fluid is incompressible.

II. LAURENT'S THEOREM

Let f(z) be analytic in a domain containing two concentric circles C_1 and C_2 with center Z_0 and the annulus between them. Then f(z) can be represented by the Laurent series

$$f(z) = \sum_{n=0}^{\alpha} a_n (z - z_0)^n + \sum_{n=1}^{\alpha} \frac{b_n}{(z - z_0)^n} \quad (1)$$

$$= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

$$+ \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots$$

Consisting of nonnegative and negative powers. The coefficients of this Laurent series are given by the integrals

$$a_n = \frac{1}{2\pi i}\oint_c \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz,$$

$$b_n = \frac{1}{2\pi i}\oint_c (z^* - z_0)^{n-1} f(z^*) dz^* \quad (2)$$

Takencounter clockwise around any simple closed path C that lies in the annulus and encircles the inner circle.

III. RESIDUE THEOREM

Let f(z) be analytic inside a simple closed path C and on C, except for finitely many singular points z_1, z_2, ..., z_k inside C. Then the integral of f(z) taken counterclockwise around C equals $2\pi i$ times the sum of the residues of f(z) at $z_1, z_2, \dots, z_k$.

$$\oint_c f(z) dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z) \quad (3)$$

IV. CIRCLE THEOREM

Suppose there is irrotational two-dimensional motion of an incompressible inviscid fluid, having no rigid boundaries, in the z-plane given by a complex potential f(z), where the singularities of f(z) are at a generator distance than a from the origin. If a circular cylinder whose cross-section C is $|z| = a$ is introduced into the fluid and held fixed, then the complex potential becomes

$$W = f(z) + \bar{f}\left(\frac{a^2}{z}\right) \quad (4)$$

V. DOUBLET

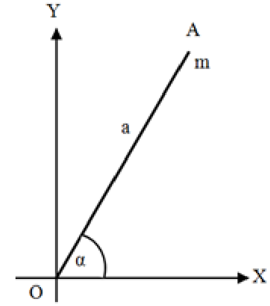

Figure 1.Doublet

Suppose there is a source of strength m at the point A, $z_0 = ae^{i\alpha}$, and a sink of equal and opposite strength at the origin,

Then OA = a, $z_0 = ae^{i\alpha}$

For the system of sources,

$$W = m \log z - m \log \left(z - ae^{i\alpha}\right)$$

$$= -m[\log\left(z - ae^{i\alpha}\right) - \log z]$$

$$= -m \log \frac{z - ae^{i\alpha}}{z}$$

$$= -m \log \left[1 - \frac{ae^{i\alpha}}{z}\right] \quad (5)$$

Using logarithmic series, we get

$$W = -m\left[-\frac{ae^{i\alpha}}{z} - \frac{a^2e^{2i\alpha}}{2z^2} - \frac{a^3e^{3i\alpha}}{3z^3} - \cdots\right]$$

$$= \frac{mae^{i\alpha}}{z} + \frac{ma^2e^{2i\alpha}}{2z^2} + \frac{ma^3e^{3i\alpha}}{3z^3} + \ldots$$

Then $W = \frac{\mu e^{i\alpha}}{z} + \frac{\mu ae^{2i\alpha}}{2z^2} + \frac{ua^2e^{3i\alpha}}{3z^3} + \ldots$ (6)

Taking $a \to 0$ and $m \to \alpha$, so that μ = constant we get the complex potential for a doublet at the origin z = 0 as $w = \frac{\mu e^{i\alpha}}{z}$.

VI. BLASIU"S THEOREM (THRUST ON A CYLINDER)

An incompressible fluid moves steadily and irrationally under no external forces parallel to the z-pane past a fixed cylinder whose section in that plane is bounded by a closed curve C. The complex potential for the flow is w. Then the action of the fluid pressure on the cylinder is equivalent to a force per unit length having components (X,Y) and a couple per unit length of moment M, where

$$Y + iX = -\frac{P}{2}\oint_c \left[\frac{dw}{dz}\right]^2 dz;$$

$$M = Re\left\{-\frac{P}{2}\oint_c z\left[\frac{d_w}{d_z}\right]^2 d_z\right\} \ldots (7)$$

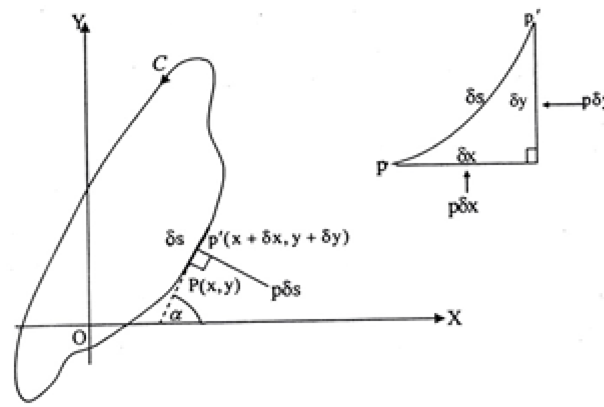

Figure 2. Thrust on a cylinder

The above figure shows the section C of the cylinder in the plane XOY. PP' is an arc elements of C of length δs. Then if p denotes the pressure at P,. the force on unit length of the section δs is $p\,\delta s$ normal to C. If α is the inclination to OX of the tangent at P to C, then x and y components of this force are $-p\delta y$ and $+p\delta x$,

where $\delta x = \delta s \cos\alpha$ and $\delta y = \delta s \sin\alpha$.

Hence $X = -\oint_c p\,dy$, $Y = \oint_c p\,dx$.

So that $Y + iX = \oint_c p\left(dx - idy\right)$.

From Bernoulli's equation $\frac{P}{\rho} + \frac{1}{2}q^2 = c$,

c = constant we obtain, $P = \rho c - \frac{1}{2}\rho q^2$,

where q is the fluid velocity, p the pressure, and ρ density.

And so $Y + iX = \rho c\oint_c\left(dx - idy\right) - \frac{\rho}{2}\oint_c q^2\left(dx - idy\right)$

$$= -\frac{\rho}{2}\oint_c q^2\left(\frac{dx}{ds} - i\frac{dy}{ds}\right) ds, \text{ since } \rho c\oint_c\left(dx - idy\right) = 0$$

$$= -\frac{\rho}{2}\oint_c q^2\left(\cos\alpha - \sin\alpha\right) ds$$

$$= -\frac{\rho}{2}\oint_c q^2 e^{-i\alpha} ds$$

But $\frac{dw}{dz} = -u + iv = -qe^{-i\alpha}$

$$q^2 = e^{2i\alpha}\left[\frac{dw}{dz}\right]^2 \text{ and so}$$

$$Y + iX = -\frac{\rho}{2}\oint_c \left[\frac{dw}{dz}\right]^2 e^{i\alpha} ds.$$

Since $dz = dx + idy = (\cos\alpha + i\sin\alpha)\, ds = e^{i\alpha} ds$,

The moment about O of the force components

where $\delta M = px\delta x + py\delta y$, so that total moment about O is

$[-p\delta y,\ p\delta x]$ is δM,

$$M = \oint_c p(xdx + ydy)$$

$$M = \oint_c \left(\rho c - \frac{1}{2}\rho q^2\right)(xdx + ydy)$$

$$M = -\frac{\rho}{2}\oint_c q^2(xdx + ydy).$$

$$\text{Now}\left[\frac{dw}{dz}\right]^2 dz = (x + i\,y)q^2 e^{-2i\alpha}(dx + i\,dy)$$

$$= (x + i\,y)q^2 e^{-2i\alpha} e^{i\alpha} ds$$

$$= q^2(x + i\,y)e^{-i\alpha} ds$$

$$= q^2(x + i\,y)\ (\cos\alpha - i\sin\alpha)ds$$

$$= q^2(x + i\,y)\ (dx - i\,dy)$$

$$= q^2(xdx + ydy) + iq^2(ydx - xdy)$$

(8)

Hence from Equation (7) and (8), we have

$$M = \text{Re}\left\{-\frac{\rho}{2}\oint_c z\left[\frac{dw}{dz}\right]^2 dz\right\} \quad (9)$$

Evaluation of the integrals for Y + iX and for M is effected using residue calculus.

VII. A SOURCE OUTSIDE A CYLINDER

A. Without circulation around the cylinder

We consider a source of strength m at a distance f from the centre of the cylinder $|z| = a\ (f > a)$.

Then the complex potential due to the source without boundary is given by

$$f(z) = -m\log(z - f) \quad (10)$$

By circle theorem, the complex potential with boundary $|z| = a$.

$$w = f(z) + \bar{f}\left[\frac{a^2}{z}\right]$$

$$= -m\log(z-f) - m\log\left[\frac{a^2}{z} - f\right]$$

$$= -m\log(z-f) + m\log z - m\log\left[z - \frac{a^2}{f}\right]$$

Neglecting constant term $-m\log(-f)$].

Then differentiating with respect to z, we get

$$\frac{dw}{dz} = -\frac{m}{z-f} + \frac{m}{z} - \frac{m}{z - \frac{a^2}{f}}$$

Squaring both sides, we get

$$\left[\frac{dw}{dz}\right]^2 = m^2\left[\frac{1}{(z-f)^2} + \frac{1}{z^2} - \frac{1}{\left(z - \frac{a^2}{f}\right)^2} - \frac{2}{(z-f)z} + \frac{2}{(z-f)\left(z - \frac{a^2}{f}\right)} - \frac{2}{z\left(z - \frac{a^2}{f}\right)}\right]$$

If the above expression is put into partial fractions, we obtain

$$-\frac{2}{(z-f)z} + \frac{2}{(z-f)\left(z - \frac{a^2}{f}\right)} - \frac{2}{z\left(z - \frac{a^2}{f}\right)}$$

$$= \frac{-\frac{2}{f}}{z-f} + \frac{\frac{2}{f}}{z} + \frac{\frac{2f}{f^2 - a^2}}{z-f} - \frac{\frac{2f}{f^2 - a^2}}{z - \frac{a^2}{f}} + \frac{\frac{2f}{a^2}}{z} - \frac{\frac{2f}{a^2}}{z - \frac{a^2}{f}}$$

Therefore the sum of residues of $\left[\frac{dw}{dz}\right]^2$ at z = 0

and $z = \frac{a^2}{f}$ is

$$m^2\left[\frac{2}{f} - \frac{2f}{f^2 - a^2} + \frac{2f}{a^2} - \frac{2f}{a^2}\right] = -\frac{2m^2a^2}{f(f^2 - a^2)}$$

Using the theorem of Blasius,

$$Y + iX = -\frac{\rho}{2}\oint_c \left[\frac{dw}{dz}\right]^2 dz$$

$$= -\frac{\rho}{2} 2\pi i \left[-\frac{2m^2a^2}{f(f^2-a^2)}\right]$$

Therefore Y = 0 and $X = \frac{2\pi\rho m^2 a^2}{f(f^2-a^2)} i$

B. With circulation around the cylinder

We shall consider here a source of strength m which is situated at the point z = c on the real axis

outside the circular cylinder $|z| = a$ and there is a circulation

of a circulation of strength $2\pi k$ around the cylinder.

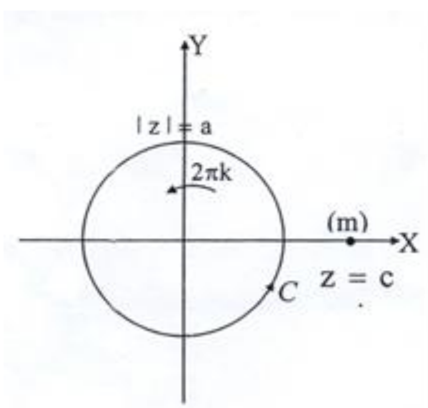

Figure 3. With Circulation around the Cylinder

Then, by Circle theorem, we obtain

$$w = f(z) + \bar{f}\left[\frac{a^2}{z}\right] + ik\log\left[\frac{z}{a}\right].$$

$$= -m\log(z-c) - m\log\left[\frac{a^2}{z}-c\right] + ik\log\left[\frac{z}{a}\right].$$

Then

$$\frac{dw}{dz} = -\left[\frac{m}{z-c} + \frac{m}{\frac{a^2}{z}-c}\left(-\frac{a^2}{z^2}\right) - \frac{ik}{\frac{z}{a}}\frac{1}{a}\right]$$

$$= -\left[\frac{m}{z-c} + \frac{ma^2}{cz\left(z-\frac{a^2}{c}\right)} \; \frac{ik}{z}\right]$$

Therefore

$$\left(-\frac{dw}{dz}\right)^2 = -\frac{m^2}{(z-c)^2} + \frac{m^2}{\left(z-\frac{a^2}{c}\right)^2} + \frac{(m+ik)^2}{z^2} + \frac{2m^2}{(z-c)\left(z-\frac{a^2}{c}\right)}$$

$$-\frac{2m(m+ik)}{(z-c)z} - \frac{2m(m+ik)}{(z-\frac{a^2}{c})z}$$

and hence

$$\left(\frac{dw}{dz}\right)^2 = \frac{m^2}{(z-c)^2} + \frac{m^2}{\left(z-\frac{a^2}{c}\right)^2} + \frac{(m+ik)^2}{z^2}$$

$$+2m^2\left[\frac{\frac{c}{c^2-a^2}}{z-c} + \frac{\frac{c}{a^2-c^2}}{z-\frac{a^2}{c}}\right] - 2m(m+ik)\left[\frac{\frac{1}{c}}{z-c} - \frac{\frac{1}{c}}{z}\right]$$

$$-2m(m+ik)\left[\frac{\frac{c}{a^2}}{z-\frac{a^2}{c}} - \frac{\frac{c}{a^2}}{z}\right].$$

By Blasius' theorem, we get

$$Y + iX = -\frac{\rho}{2}\oint_c \left[\frac{dw}{dz}\right]^2 dz$$

$$= -\frac{\rho}{2} 2\pi i \left[\text{sum of residues of } \left[\frac{dw}{dz}\right]^2 \text{ at } z = 0 \text{ and } z = \frac{a^2}{c}\right]$$

$$= -\pi\rho i\left[\frac{2m^2c}{a^2-c^2} + \frac{2m(m+ik)}{c} + \frac{2m(m+ik)c}{a^2} + \frac{2m(m+ik)c}{a^2}\right]$$

$$Y + iX = -2\pi\rho m^2 i\left[-\frac{c}{c^2-a^2} + \frac{1}{c}\right] + \frac{2\pi\rho mk}{c}$$

$$= -2\pi\rho m^2 i\left[\frac{-c^2+c^2-a^2}{c\left(c^2-a^2\right)}\right] + \frac{2\pi\rho mk}{c}$$

$$= \frac{2\pi\rho m^2 a^2}{c\left(c^2 - a^2\right)} i + \frac{2\pi\rho mk}{c}.$$

Therefore we obtain $Y = \frac{2\pi\rho mk}{c}$ and $X = \frac{2\pi\rho m^2 a^2}{c\left(c^2 - a^2\right)}$.

If θ is the angle made by the resultant thrust with the positive x axis, it is given by
\

$$\theta = \tan^{-1}\frac{Y}{X}$$

$$\theta = \tan^{-1}\left[\frac{2\pi\rho mk}{c}\frac{c\left(c^2 - a^2\right)}{2\pi\rho m^2 a^2}\right]$$

$$\theta = \tan^{-1}\left[\frac{k\left(c^2 - a^2\right)}{ma^2}\right]$$

VI. A DOUBLET OUTSIDE A CYCLINDER

Let a doublent of strength μ be at a distance f from the

centre of the cylinder $|z| = a.(f > a)$.

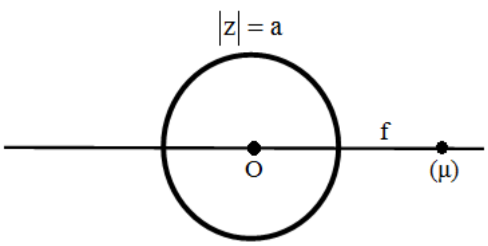

Figure 4. Doublet Outside a Cylinder

Then the complex potential due to the doublet at z = f is

$$f(z) = \frac{\mu}{z - f}.$$

By circle theorem, the complex potential

due to the doublet with boundary $|z| = a$ is given by

$$w = f(z) + \bar{f}\left[\frac{a^2}{z}\right]$$

$$w = \frac{\mu}{z - f} + \frac{\mu}{\frac{a^2}{z} - f}$$

$$w = \frac{\mu}{z - f} + \frac{\mu z}{-f\left(z - \frac{a^2}{f}\right)}$$

$$w = \frac{\mu}{z - f} + \frac{\mu}{-f} - \frac{\mu a^2}{f^2} - \frac{1}{z - \frac{a^2}{f}}$$

$$w = \frac{\mu}{z - f} + -\frac{\frac{\mu a^2}{f^2}}{z - \frac{a^2}{f}}, \text{ neglecting constant} -\frac{\mu}{f}$$

Hence, $\frac{dw}{dz} = \frac{\mu}{z - f} + -\frac{\frac{\mu a^2}{f^2}}{z - \frac{a^2}{f}}$

Squaring and expressing the result inpartial fractions,

we obtain

$$\left[\frac{dw}{dz}\right]^2 = \frac{\mu^2}{(z - f)^4} + -\frac{\mu^2\frac{a^4}{f^4}}{\left(z - \frac{a^2}{f}\right)^4}$$

$$-2\mu^2\frac{a^2}{f^2}\left[\frac{-2\frac{f^3}{\left(f^2 - a^2\right)^3}}{z - f} + \frac{\frac{f^2}{\left(f^2 - a^2\right)^2}}{(z - f)^2}\right.$$

$$\left. + \frac{\frac{2f^2}{\left(f^2 - a^2\right)^3}}{z - \frac{a^2}{f}} + \frac{\frac{f^2}{\left(f^2 - a^2\right)^2}}{\left(z - \frac{a^2}{f}\right)^2}\right.$$

By Boasiu's theorem we obtain,

$$Y + iX = -\frac{\rho}{2}\oint_c\left[\frac{dw}{dz}\right]^2 dz$$

$$Y + iX = -\frac{\rho}{2}\times 2\pi i\left\{\text{sum of residues of }\left[\frac{dw}{dz}\right]^2 \text{ at } z = \frac{a^2}{f}\right\}$$

$$Y + iX = -\pi\rho i\left[-\frac{2\mu^2 a^2}{f^2}\left(\frac{2f^3}{\left(f^2 - a^2\right)^3}\right)\right]$$

$$Y + iX = \frac{4\pi\rho\mu^2 a^2 f}{\left(f^2 - a^2\right)^3} i$$

Therefore Y=0 and X = $\frac{4\pi\rho\mu^2 a^2 f}{\left(f^2 - a^2\right)^3}$ (26)

VIII. CONCLUSION

Finally, this paper is concluded that, Blasiu's theorem stated that an incompressible fluid is moved by a steady irrotational fluid motion under no external force. Blasiu's theorem is expressed by the action of the fluid pressure on the cylinder.

The Blasiu's theorem gives a convenient formula for the force on a two- dimensional body in an incompressible potential flow field. The direct way to find the force on the body is to integrate the pressure forces over the surface.

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