Signed Unidominating Functions of Corona Product Graph $P_n \odot K_{1,m}$

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Abstract- In recent years Graph Theory has been realised as one of the most developed branches of Mathematics with wide applications to several fields - like computer science, social sciences, Science & Technology, etc. The concept of 'Dominating function' in domination theory arouse much interest of research which has emerged rapidly in the last three decades. An introduction and an extensive overview on domination in graphs and related topics is surveyed and detailed in the two books by Haynes et al. [5, 6].

Corona product graphs is a new concept introduced by Frucht and Harary [2] has become an interesting area of research at present. The object is to construct a new and simple operation on two graphs G_1 and G_2 called their corona, denoted by $G_1 \odot G_2$ with the property that the group of the new graph is in general isomorphic with the wreath product of the groups of G_1 and of G_2.

A new concept signed unidominating function is introduced and studied this concept for corona product graph $P_n \odot K_{1,m}$ and determined the signed unidomination and upper signed unidomination numbers for $P_n \odot K_{1,m}$. Also the number of signed unidominating functions of minimum weight is found.

Indexed Terms- Signed unidominating function, signed unidomination number, minimal signed unidominating function, upper signed unidomination number.

I. INTRODUCTION

In recent years dominating functions in domination theory playing a key role as they have interesting applications. The concepts of dominating functions are introduced by Hedetniemi [3]. Anantha Lakshmi [1] has introduced new concepts unidomination, upper unidomination, and minimal unidominating function of a graph and studied these functions for some standard graphs.

A new product on two graphs G_1 and G_2, called corona product denoted by $G_1 \odot G_2$, was introduced by Frucht and Harary [2]. The concept signed dominating function of corona product graph $P_n \odot K_{1,m}$ was studied by Siva Parvathi [4].

The authors have introduced new concepts signed unidominating function and upper signed unidomination number of a graph in this paper and this is studied for corona product graph $P_n \odot K_{1,m}$. Also the signed unidomination and upper signed unidomination number of the above said graph is found. Further, the number of signed unidominating functions of minimum weight and minimal signed unidominating functions of maximum weight for this graph are determined.

II. CORONA PRODUCT OF P_n AND $K_{1,m}$

The corona product of a path P_n with a star graph $K_{1,m}$ is a graph obtained by taking one copy of a n – vertex graph P_n and n copies of $K_{1,m}$ and then joining the i^{th} - vertex of P_n to every vertex of i^{th} - copy of $K_{1,m}$. This is denoted by $P_n \odot K_{1,m}$.

III. SIGNED UNIDOMINATION NUMBER OF $P_n \odot K_{1,m}$

In this section the concepts of signed unidominating function, signed unidomination number are defined. The signed unidomination number and the number of signed unidominating functions of minimum weight of $P_n \odot K_{1,m}$ are determined.

Definition 1: Let $G(V,E)$ be a connected graph. A function $f: V \to \{-1,1\}$ is said to be a signed unidominating function if

$$\sum_{u \in N[v]} f(u) \geq 1 \quad \forall\, v \in V \ and\ f(v) = 1,$$

and $$\sum_{u \in N[v]} f(u) = 1 \quad \forall\, v \in V \ \ and \ \ f(v) = -1.$$

Definition 2: The signed unidomination number of a graph $G(V,E)$ is defined as

$\min\{f(V)/f$ is a signed unidominating function$\}$.

It is denoted by $\gamma_{su}(G)$.

Here $f(V) = \sum_{u \in V} f(u)$ is called as the weight of the signed unidominating function f.

That is the signed unidomination number of a graph $G(V,E)$ is the minimum of the weights of the signed unidominating functions of G.

Theorem 3.1: The signed unidomination number of $P_n \odot K_{1,m}$ is

$$\begin{cases} 2n & when\ m\ is\ even, \\ n & when\ m\ is\ odd. \end{cases}$$

Proof: Let $P_n \odot K_{1,m}$ be the given corona product graph.

To find the signed unidomination number of $P_n \odot K_{1,m}$ the following cases arises.

Case 1: Suppose m is even.

Define a function $f: V \to \{-1,1\}$by

$$f(v_i) = 1 \text{ for } i = 1,2, \dots\dots, n,$$

$f(u_i) = 1$ for$i = 1,2, \dots\dots, n$

and$f(u_{ij}) =$

$$\begin{cases} -1 & for\ \frac{m}{2}\ vetices\ in\ each\ copy\ of\ K_{1,m}, \\ 1 & otherwise. \end{cases}$$

Now we show that f is a signed unidomonating function.

Let $i \neq 1 and\ i \neq n$.

If $v_i \in P_n$ then

$$\sum_{u \in N[v_i]} f(u) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) + f(u_i) + f(u_{i1}) + f(u_{i2}) + \dots + f(u_{im}) = 1+1+1+1 + \left[\left(\frac{m}{2}\right)(-1) + \left(\frac{m}{2}\right)(1)\right] = 4.$$

For $i = 1$, if $v_1 \in P_n$ then

$$\sum_{u \in N[v_1]} f(u) = f(v_1) + f(v_2) + f(u_1) + f(u_{11}) + f(u_{12}) + \dots + f(u_{1m}) = 1+1+1 + \left[\left(\frac{m}{2}\right)(-1) + \left(\frac{m}{2}\right)(1)\right] = 3.$$

For $i = n$, if $v_n \in P_n$ then

$$\sum_{u \in N[v_n]} f(u) = f(v_{n-1}) + f(v_n) + f(u_n) + f(u_{n1}) + f(u_{n2}) + \dots + f(u_{nm}) = 1+1+1 + \left[\left(\frac{m}{2}\right)(-1) + \left(\frac{m}{2}\right)(1)\right] = 3.$$

If $u_i \in K_{1,m}$ then

$$\sum_{u \in N[u_i]} f(u) = f(v_i) + f(u_i) + f(u_{i1}) + f(u_{i2}) + \dots + f(u_{im}) = 1 + 1 + \left[\left(\frac{m}{2}\right)(-1) + \left(\frac{m}{2}\right)(1)\right] = 2.$$

If $u_{ij} \in K_{1,m}$ then $f(u_{ij}) = 1$ or $f(u_{ij}) = -1$.

Let $u_{ij} \in K_{1,m}$ and $f(u_{ij}) = 1$. Then

$$\sum_{u \in N[u_{ij}]} f(u) = f(v_i) + f(u_i) + f(u_{ij}) = 1+1+1 = 3.$$

Let $u_{ij} \in K_{1,m}$and $f(u_{ij}) = -1$. Then

$$\sum_{u \in N[u_{ij}]} f(u) = f(v_i) + f(u_i) + f(u_{ij}) = 1 + 1 + (-1) = 1$$

Hence it follows that f is a signed unidominating function.

$$\text{Now } f(V) = \sum_{u \in P_n} f(u) + \sum_{u \in K_{1,m}} f(u) = \underbrace{(1+1+1\dots+1)}_{(n-times)} +$$

$$\underbrace{\left\{1 + \underbrace{((-1)+\dots(-1))}_{\left(\frac{m}{2}-times\right)} + \underbrace{(1+\dots+1)}_{\left(\frac{m}{2}-times\right)}\right\} + \dots + \left\{1 + \underbrace{((-1)+\dots(-1))}_{\left(\frac{m}{2}-times\right)} + \underbrace{(1+\dots+1)}_{\left(\frac{m}{2}-times\right)}\right\}}_{(n-\text{times})}$$

$= n + \left[n + \left(\frac{m}{2}\right)(-n) + \left(\frac{m}{2}\right)(n) \right] = 2n.$

Thus $f(V) = 2n.$

Now for all other possibilities of assigning values 1 and -1 to the vertices of P_n and vertex u_i and vertices u_{ij} in each copy of $K_{1,m}$, we can see that the resulting functions are not signedunidominating functions.

Thus the function defined above is the only signed unidominating function.

Therefore $\gamma_{su}\left(P_n \odot K_{1,m}\right) = 2n$ when m is even.

Case 2: Suppose m is odd.

Define a function $f: V \to \{-1,1\}$ by

$f(v_i) = 1$ for $i = 1,2, \dots \dots, n,$

$f(u_i) = 1$ for $i = 1,2, \dots \dots, n$

and $f(u_{ij}) =$

$$\begin{cases} -1 & for\ \frac{m+1}{2}\ vetices\ in\ each\ copy\ of\ K_{1,m}, \\ 1 & for\ \frac{m-1}{2}\ vetices\ in\ each\ copy\ of\ K_{1,m}. \end{cases}$$

Now we check that f is a signed unidomonating function.

Let $i \neq 1 and\ i \neq n.$

If $v_i \in P_n$ then

$$\sum_{u \in N[v_i]} f(u) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) + f(u_i) + f(u_{i1}) + f(u_{i2}) + \cdots + f(u_{im}) = 1 + 1 + 1 + 1 + \left[\left(\frac{m+1}{2}\right)(-1) + \left(\frac{m-1}{2}\right)(1)\right] = 3.$$

Let $i = 1.$

If $v_1 \in P_n$ then

$$\sum_{u \in N[v_1]} f(u) = f(v_1) + f(v_2) + f(u_1) + f(u_{11}) + f(u_{12}) + \cdots + f(u_{1m}) = 1 + 1 + 1 + \left[\left(\frac{m+1}{2}\right)(-1) + \left(\frac{m-1}{2}\right)(1)\right] = 2.$$

For $i = n$ we have

If $v_n \in P_n$ then

$$\sum_{u \in N[v_n]} f(u) = f(v_{n-1}) + f(v_n) + f(u_n) + f(u_{n1}) + f(u_{n2}) + \cdots + f(u_{nm}) = 1 + 1 + 1 + \left[\left(\frac{m+1}{2}\right)(-1) + \left(\frac{m-1}{2}\right)(1)\right] = 2.$$

If $u_i \in K_{1,m}$ then

$$\sum_{u \in N[u_i]} f(u) = f(v_i) + f(u_i) + f(u_{i1}) + f(u_{i2}) + \cdots + f(u_{im}) = 1 + 1 + \left[\left(\frac{m+1}{2}\right)(-1) + \left(\frac{m-1}{2}\right)(1)\right] = 1.$$

If $u_{ij} \in K_{1,m}$ then $f(u_{ij}) = 1$ or $f(u_{ij}) = -1.$

Let $u_{ij} \in K_{1,m}$ and $f(u_{ij}) = 1.$ Then

$$\sum_{u \in N[u_{ij}]} f(u) = f(v_i) + f(u_i) + f(u_{ij}) = 1 + 1 + 1 = 3.$$

Let $u_{ij} \in K_{1,m}$ and $f(u_{ij}) = -1.$ Then

$$\sum_{u \in N[u_{ij}]} f(u) = f(v_i) + f(u_i) + f(u_{ij}) = 1 + 1 + (-1) = 1.$$

$$\text{Now } f(V) = \sum_{u \in P_n} f(u) + \sum_{u \in K_{1,m}} f(u) = \underbrace{(1 + 1 + 1 \dots + 1)}_{(n-times)} + \underbrace{\left\{1 + \underbrace{((-1) + \cdots (-1))}_{\left(\frac{m+1}{2}-times\right)} + \underbrace{(1 + \cdots + 1)}_{\left(\frac{m-1}{2}-times\right)}\right\} + \cdots + \left\{1 + \underbrace{((-1) + \cdots (-1))}_{\left(\frac{m+1}{2}-times\right)} + \underbrace{(1 + \cdots + 1)}_{\left(\frac{m-1}{2}-times\right)}\right\}}_{(n-\text{times})}$$

$$= n + \left[n + \left(\frac{m+1}{2}\right)(-n) + \left(\frac{m-1}{2}\right)(n)\right] = n.$$

Thus $f(V) = n.$

Now for all other possibilities of assigning values 1 and -1 to the vertices of P_n and vertex u_i and vertices u_{ij} in each copy of $K_{1,m}$, we can see that the resulting functions are not signedunidominating functions.

Therefore the function defined above is the only signed unidominating function.

Therefore $\gamma_{su}\left(P_n \odot K_{1,m}\right) = n$ when m is odd.

Theorem 3.2: If m is even then the number of signed unidominating functions of $P_n \odot K_{1,m}$ with minimum weight $2n$ is 1 and if m is odd then the number of signed unidominating functions of $P_n \odot K_{1,m}$ with minimum weight n is 1.

Proof: Follows by Theorem 3.1.

IV. UPPER SIGNED UNIDOMINATION NUMBER OF $P\odot K_{1,m}$

In this section the concepts of minimal signed unidominating function, upper signed unidomination number are defined. The upper signed unidomination number and the number of minimal signed unidominating functions of maximum weight of $P_n \odot K_{1,m}$ are determined. Further the results obtained are illustrated.

Definition 1: Let f and g be functions from V to $\{-1,1\}$. We say that $g < f$ if $g(u) \le f(u)\ \forall\, u \in V,$ with strict inequality for at least one vertex u.

Definition 2: A signed unidominating function $f: V \to \{-1,1\}$ is called a minimal signed unidominating function if for all $g < f, g$ is not a signed unidominating function.

Definition 3: The upper signed unidomination number of a graph $G(V, E)$ is defined as

max $\{f(V)/f$ is a minimal signed unidominating function$\}$.

It is denoted by $\Gamma_{su}(G)$.

Theorem 4.1: The upper signed unidomination number of $P_n \odot K_{1,m}$ is

$$\begin{cases} 2n & when\ m\ is\ even, \\ n & when\ m\ is\ odd. \end{cases}$$

Proof: Let $P_n \odot K_{1,m}$ be the given graph.

Case 1: Suppose m is even.

Define a function $f: V \to \{-1,1\}$by

$$f(v_i) = 1 \text{ for } i = 1,2, \dots \dots, n,$$

$f(u_i) = 1$ for$i = 1,2, \dots \dots, n$

and$f(u_{ij}) =$

$$\begin{cases} -1 & for\ \frac{m}{2}\ vetices\ in\ each\ copy\ of\ K_{1,m}, \\ 1 & otherwise. \end{cases}$$

This function is same as the function defined in Case 1 of Theorem 3.1 and it is shown that $f$is a signed unidominating function.

Now we check for the minimality of f.

Define a function $g: V \to \{-1,1\}$by

$$g(v_i) = \begin{cases} -1 & for\ v_i = v_k \in P_n\ for\ some\ k \\ 1 & otherwise \end{cases}$$

and$g(u_i) = 1,\ u_i \in K_{1,m}$ for $i = 1,2, \dots \dots, n$

and$g(u_{ij}) =$

$$\begin{cases} -1 & for\ \frac{m}{2}\ vetices\ in\ each\ copy\ of\ K_{1,m}, \\ 1 & otherwise \end{cases}$$

for$i = 1,2, \dots \dots, n$ and j$= 1,2, \dots \dots, m$.

Then by the definition of g it is obvious that $g < f$.

Suppose $i = k$. Then $g(v_k) = -1$.

For $v_k \in P_n$ we have

$$\sum_{u \in N[v_k]} g(u) = g(v_{k-1}) + g(v_k) + g(v_{k+1}) + g(u_k) + g(u_{k1}) + g(u_{k2}) + \cdots + g(u_{km}) = 1 + (-1) + 1 + 1 + \left[\left(\frac{m}{2}\right)(-1) + \left(\frac{m}{2}\right)(1)\right] = 2 \neq 1.$$

That is the condition for signed unidominating function fails at the vertex $v_k \in P_n$.

So $\sum_{u \in N[v_k]} g(u) \neq 1,\ as\ g(v_k) = -1.$

Thus g is not a signed unidominating function.

Since g is defined arbitrarily, it follows that there exists no $g < f$ such that g is a signed unidominating function.

For all other possibilities of defining a function $g < f$, we can see that g is not a signed unidominating function.

Hence f is a minimal signed unidominating function.

Further f is the only one minimal signed unidominating function because any other possible assignment of values $-1, 1$ to the vertices of P_n and $K_{1,m}$ does not make f no more a signed unidominating function.

$$\text{Now } f(V) = \sum_{u \in P_n} f(u) + \sum_{u \in K_{1,m}} f(u)$$

$$= \underbrace{(1 + 1 + 1 \dots + 1)}_{(n-times\ )} + \underbrace{\left\{1 + \underbrace{((-1) + \cdots (-1))}_{\left(\frac{m}{2}-times\ \right)} + \underbrace{(1 + \cdots + 1)}_{\left(\frac{m}{2}-times\ \right)}\right\} + \cdots + \left\{1 + \underbrace{((-1) + \cdots (-1))}_{\left(\frac{m}{2}-times\ \right)} + \underbrace{(1 + \cdots + 1)}_{\left(\frac{m}{2}-times\ \right)}\right\}}_{(n-\text{times}\ )}$$

$$= n + \left[\, n + \left(\frac{m}{2}\right)(-n) + \left(\frac{m}{2}\right)(n)\right] = 2n.$$

Thus $f(V) = 2n$.

Now

max $\{f(V)/f$ is a minimal signed unidominating function$\} = 2n$, because f is the only one minimal signed unidominating function.

Therefore $\Gamma_{su}(P_n \odot K_{1,m}) = 2n$.

Case 2: Suppose m is odd.

Define a function $f: V \to \{-1, 1\}$by

$$f(v_i) = 1 \text{ for } i = 1,2, \dots \dots, n,$$

$f(u_i) = 1$ for$i = 1,2, \dots \dots, n$

and$f(u_{ij}) =$
$\begin{cases} -1 \ \ for \ \frac{m+1}{2} \ vetices \ in \ each \ copy \ of \ K_{1,m}, \\ 1 \ \ for \ \frac{m-1}{2} \ vetices \ in \ each \ copy \ of \ K_{1,m}. \end{cases}$

This function is same as the function defined in Case 2 of Theorem 3.1 and it is shown that f is a signed unidominating function.

Now we check for the minimality of f.

Define a function $g: V \rightarrow \{-1,1\}$by

$g(v_i) =$
$\begin{cases} -1 \ for \ v_i = \ v_k \in P_n \ for \ some \ k, \\ 1 \qquad\qquad otherwise \end{cases}$

and$g(u_i) = 1, \ u_i \in K_{1,m}$ for $i = 1,2, \dots \dots, n$

and$g(u_{ij}) =$
$\begin{cases} -1 \ \ for \ \frac{m+1}{2} \ vetices \ in \ each \ copy \ of \ K_{1,m}, \\ 1 \ for \ \frac{m-1}{2} \ vetices \ in \ each \ copy \ of \ K_{1,m} \end{cases}$

for$i = 1,2, \dots \dots, n$ and j$= 1,2, \dots \dots, m$.

Then by the definition of g it is obvious that $g < f$.

Suppose $i = k$. Then $g(v_k) = -1$.

For $u_k \in K_{1,m}$, we have

$$\sum_{u \in N[u_k]} g(u) = g(v_k) + g(u_k) + g(u_{k1}) + g(u_{k2}) + \dots + g(u_{km}) = (-1) + 1 + \left[\left(\frac{m+1}{2}\right)(-1) + \left(\frac{m-1}{2}\right)(1)\right] = -1 \neq 1.$$

That is the condition for signed unidominating function fails at the vertex $u_k \in K_{1,m}$.

So $\sum_{u \in N[u_k]} g(u) \neq 1$, as $g(v_k) = -1$.

Thus g is not a signed unidominating function.

Since g is defined arbitrarily, it follows that there exists no $g < f$ such that g is a signed unidominating function.

For all other possibilities of defining a function $g < f$, we can see that g is not a signed unidominating function.

Hence f is a minimal signedunidominating function.

Further f is the only one minimal signed unidominating function because any other possible assignment of values $-1, 1$ to the vertices of P_n and $K_{1,m}$ does not make f no more a signed unidominating function.

$$\text{Now } f(V) = \sum_{u \in P_n} f(u) + \sum_{u \in K_{1,m}} f(u)$$
$$= \underbrace{(1 + 1 + 1 \dots + 1)}_{(n-times)} +$$
$$\underbrace{\left\{1 + \underbrace{((-1) + \cdots (-1))}_{\left(\frac{m+1}{2}-times\right)} + \underbrace{(1 + \cdots + 1)}_{\left(\frac{m-1}{2}-times\right)}\right\} + \cdots + \left\{1 + \underbrace{((-1) + \cdots (-1))}_{\left(\frac{m+1}{2}-times\right)} + \underbrace{(1 + \cdots + 1)}_{\left(\frac{m-1}{2}-times\right)}\right\}}_{(n-\text{times})}$$
$$= n + \left[n + \left(\frac{m+1}{2}\right)(-n) + \left(\frac{m-1}{2}\right)(n)\right] = n.$$

Thus $f(V) = n$.

Now

max $\{f(V)/f$ is a minimal signed unidominating function$\} = n$, because f is the only one minimal signed unidominating function.

Therefore $\Gamma_{su}(P_n \odot K_{1,m}) = n$.

Theorem 4.2: If m is even then the number of minimal signed unidominating functions of $P_n \odot K_{1,m}$ with maximum weight $2n$is 1 and if m is odd then the number of minimal signedunidominating functions of $P_n \odot K_{1,m}$ with maximum weight n is 1.

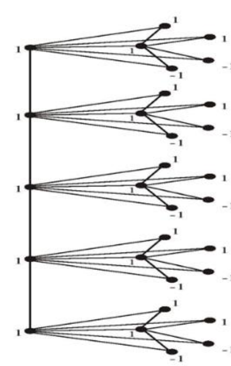
Proof: Follows by Theorem 4.1.

V. ILLUSTRATIONS

Signed unidomination number

Theorem 3.1: Case 1

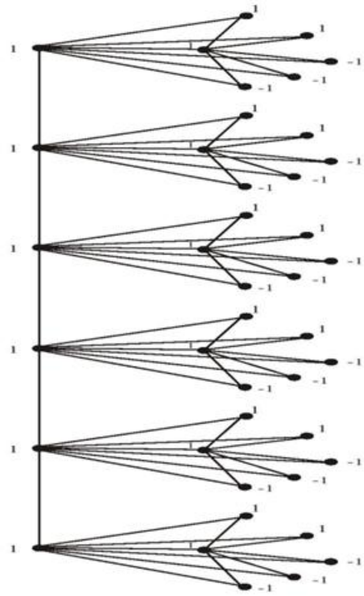
The functional values are given at each vertex of the graph $P_5 \odot K_{1,4}$.



$\gamma_{su}(P_5 \odot K_{1,4}) = 10$

Theorem 3.1: Case 2

The functional values are given at each vertex of the graph $P_6 \odot K_{1,5}$.



$\gamma_{su}\left(P_6 \odot K_{1,5}\right) = 6$

VI. CONCLUSION

It is interesting to study various graph theoretic properties and domination parameters of corona product graph of a path with a star graph. Signed unidominating functions and upper signed unidomination number of this graph are studied by the authors. Study of these graphs enhances further research and throws light on future developments.

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