Interval Graphs with Alternate Cliques of Size 3 - Roman Domination Number

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Abstract- Today graph theory is one of the most flourishing braches of modern mathematics. Graphs are useful in enhancing the understanding of the organization and behavioural characteristics of complex system. The study of domination in graphs originated around 1850 has become the source of interest to the researchers. Interval graphs form a special class of graphs with many interesting properties and revealed their practical relevance for modelling problems arising in the real world. The last 40 years have witnessed a spectacular growth of domination in interval graphs due to its wide range of applications to many fields. Roman domination in graphs is introduced by Cockayne et.al [3, 4] and they studied this concept for various graphs. In this paper a study of Roman domination in Interval graphs with alternate cliques of size 3 is carried out.

Indexed Terms- Roman dominating function, Roman domination number, Interval family, Interval graph.

I. INTRODUCTION

The theory of domination in graphs introduced by Ore [9] and Berge [2] is an emerging area of research in graph theory today. An introduction and an extensive overview on domination in graphs and related topics is surveyed and detailed in the two books by Haynes et.al. [5, 6]. Many graph theorists, to mention some of them Allan and Laskar.[1], Cockayne. and Hedetniemi [3] and others have studied various types of domination parameters of graphs.

Let $G(V,E)$ be a graph. A subset D of V is said to be a dominating set of G if every vertex in $V-D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set is called the domination number and is denoted by $\gamma(G)$.

We consider finite graphs without loops and multiple edges.

II. ROMAN DOMINATING FUNCTION

The Roman dominating function of a graph G was defined by Cockayne et.al [3, 4]. The definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart [7] entitled "Defend the Roman Empire!" and suggested even earlier by ReVelle [10]. Domination number and Roman domination number in an interval graph with consecutive cliques of size 3 are studied by C. Jaya Subba Reddy, M. Reddappa and B. Maheswari [8].
A Roman dominating function on a graph

$G(V,E)$ is a function $f: V \rightarrow \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=2$.The weight of a Roman dominating function is the value $f(V)=\sum_{v\in V} f(v)$. The minimum weight of a Roman dominating function on a graph G is called as the Roman domination number of G. It is denoted by$\gamma_R(G)$. If $\gamma_R(G) = 2\,\gamma(G)$ then G is called a Roman graph.

Let $f: V \rightarrow \{0,1,2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f where $V_i = \{v \in V / f(v) = i\}$for $i = 0,1,2$. Then there exists a 1-1 correspondence between the functions $f: V \rightarrow \{0,1,2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus we write$f = (V_0, V_1, V_2)$.

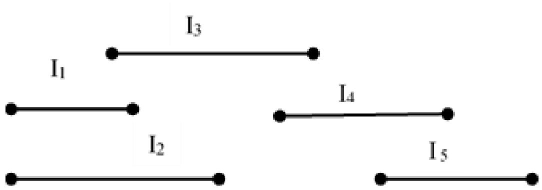
A function $f = (V_0, V_1, V_2)$ becomes a Roman dominating function if the set V_2 dominates V_0.

III. INTERVAL GRAPH

Let $I = \{I_1, I_2, I_3, \dots \dots \dots .. I_n\}$ be an interval family, where each I_i is an interval on the real line and $I_i = [a_i, b_i]$ for $i = 1, 2, 3, \dots \dots .. n$. Here a_i is called left end point and b_i is called the right end point of I_i. Without loss of generality, we assume that all end points of the intervals in I are distinct numbers between 1 and 2n. Two intervals $i = [a_i, b_i]$ and $j = [a_j, b_j]$ are said to intersect each other if either $a_j < b_i$ or $a_i < b_j$. The intervals are labelled in the increasing order of their right end points.

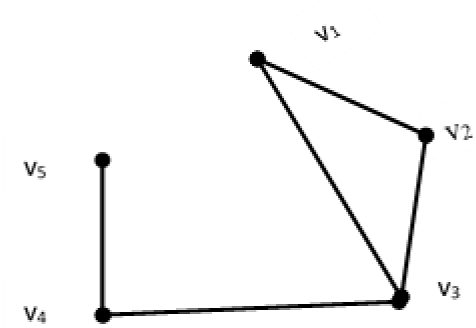
Let $G(V, E)$ be a graph. G is called an interval graph if there is a 1-1 correspondence between V and I such that two vertices of G are joined by an edge in E if and only if their corresponding intervals in I intersect. If i is an interval in I the corresponding vertex in G is denoted by v_i.

Consider the following interval family.



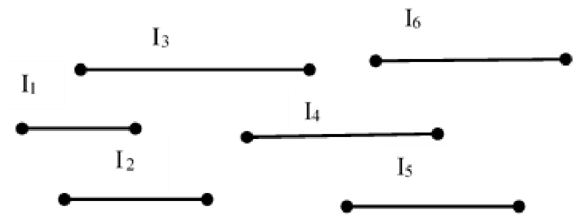
Interval family

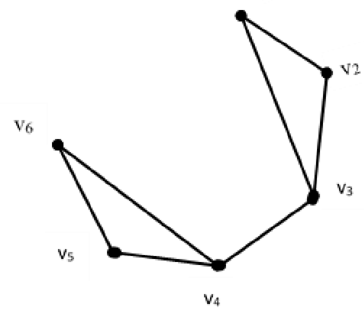
The corresponding interval graph is given by



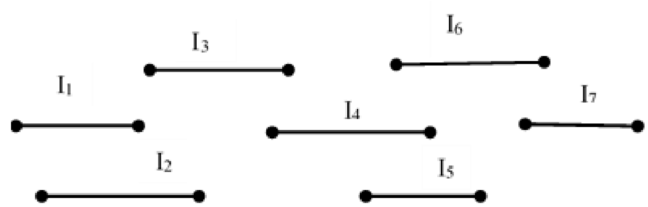
Interval graph

Consider the following interval family.



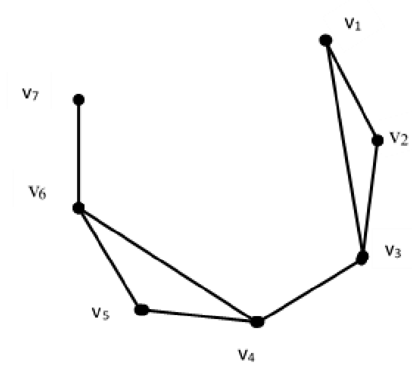


Consider the following interval family.



Interval family

The corresponding interval graph is given by



Interval graph

In what follows we consider interval graphs of this type. We observe that when n is 3k+3 this interval graph has adjacent cliques of size 3, k = 1,2,3…..When n is 3k+2 this interval graph has adjacent cliques of size 3 and the last clique is adjacent with two edges and when n is 3k+4 the last clique is adjacent with one edge, k =1,2,3….. We denote this type of interval graph by$\mathcal{G}$. The domination and Roman domination is studied in the following for the interval graph$\mathcal{G}$.

IV. RESULTS

Theorem 4.1: Let $\mathcal{G}$be the Interval graph with n vertices. Then the domination number of $\mathcal{G}$ is $\gamma(\mathcal{G}) = k + 1$ for $n = 3k + 2, 3k + 3, 3k + 4$, where $k = 1,2,3 \ldots \ldots$..respectively.

Proof: Let $\mathcal{G}$ be the Interval graph. Let D denote the dominating set of $\mathcal{G}$.

Suppose k = 1. Then $n = 5, 6, 7$. For $n = 5$ we can see that $D = \{v_3, v_5\}$ and for $n = 6$ and 7 we can see that $D = \{v_3, v_6\}$ is a dominating set of $\mathcal{G}$ respectively. Thus $\gamma(\mathcal{G}) = 2$ for $n = 5, 6, 7$.

Similar is the case for $n = 8, 9, 10$, where the dominating sets are respectively $D = \{v_3, v_6, v_8\}$, $D = \{v_3, v_6, v_9\}$, $D = \{v_3, v_6, v_9\}$ and the domination number $\gamma(\mathcal{G}) = 3$.

Again for $n = 11, 12, 13$, we see that $\gamma(\mathcal{G}) = 4$ and the dominating sets are $D = \{v_3, v_6, v_9, v_{11}\}$,$D = \{v_3, v_6, v_9, v_{12}\}$, $D = \{v_3, v_6, v_9, v_{12}\}$ respectively.

Thus$\gamma(\mathcal{G}) = 2$ for $n = 5, 6, 7$.

$= 3$ for $n = 8, 9, 10$.

$= 4$ for $n = 11, 12, 13$.

Thus generalizing, we get that the general form of dominating sets of $\mathcal{G}$ as

$D = \{v_3, v_6, v_9, v_{11} \ldots \ldots \ldots \ldots \ldots .. v_n\}$

for $n = 5, 8, 11\ 14, \ldots \ldots . \ldots \ldots$

$D = \{v_3, v_6, v_9, v_{11} \ldots \ldots \ldots \ldots \ldots .. v_n\}$

for $n = 6, 9, 12\ 15, \ldots \ldots . \ldots \ldots$

$D = \{v_3, v_6, v_9, v_{11} \ldots \ldots \ldots \ldots \ldots .. v_{n-1}\}$

for $n = 7, 10, 13, 16, \ldots \ldots . \ldots \ldots$

And$\gamma(\mathcal{G}) = k + 1$ for $n = 3k + 2, 3k + 3, 3k + 4$,respectively, where $k = 1,2,3 \ldots \ldots \ldots$

Corollary 4.2: Let$\mathcal{G}$ be the interval graph with n vertices. Then the dominating set in Theorem 4.1 becomes an independent dominating set.

Proof: Let $\mathcal{G}$ be the Interval graph. By the selection of vertices into the dominating set as in Theorem 4.1, it is obvious that they form an independent set. Hence the dominating set becomes an independent dominating set.

Theorem 4.3: Let $\mathcal{G}$ be the interval graph with n vertices, where $1 < n < 5$. Then $\gamma(\mathcal{G}) = 1$.

Proof: Let $\mathcal{G}$ be the interval graph with n vertices, where $1 < n < 5$.

Then it is clear that $\{v_2\}$ is the dominating set when n = 2, 3 and $\{v_3\}$ is the dominating set when n = 4.

That is $\gamma(\mathcal{G}) = 1$.

Theorem 4.4: The Roman domination number of an Interval graph $\mathcal{G}$ with n vertices is

$\gamma_R(\mathcal{G}) = 2k + 1$ for $= 3k + 2$,

$= 2k + 2$ for$n = 3k + 3, 3k + 4$, respectively , where $k = 1,2,3 \ldots \ldots \ldots$

Proof: Let $\mathcal{G}$ be the interval graph with n vertices. Let the vertex set of $\mathcal{G}$ be

$\{v_1, v_2, v_3, v_4 \ldots \ldots \ldots \ldots \ldots .. v_n\}$.

Case 1: Suppose $n = 3k + 2$, where $k = 1,2,3 \ldots \ldots$

Let $f : V \to \{0, 1, 2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f where $V_i = \{v \in V / f(v) = i\}$ for $i = 0, 1, 2$. Then there exist a 1-1 correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered pairs (V_0, V_1, V_2) of V. Thus we write $f = (V_0, V_1, V_2)$.

Let $V_1 = \{v_n\}$,

$V_2 = \{v_3, v_6, v_9, v_{11} \ldots \ldots \ldots \ldots \ldots .. v_{n-2}\}$,

V_0= V-$\{V_1 \cup V_2\}$= $\{v_1, v_2, v_4, \ldots \ldots \ldots \ldots \ldots .. v_{n-1}\}$.

We observe that $V_1 \cup V_2$ is a dominating set of $\mathcal{G}$ (by Theorem 4.1) and the set V_2 dominates V_0.

Therefore $f = (V_0, V_1, V_2)$ is a Roman dominating function of $\mathcal{G}$. We know that $\gamma(\mathcal{G}) = k + 1$.

So $|V_2| = k, |V_1| = 1, |V_0| = n - k - 1$.

Therefore

$$\sum_{v \in V} f(v) = \sum_{v \in V_0} f(v) + \sum_{v \in V_1} f(v) + \sum_{v \in V_2} f(v).$$

$$= 1 + 2k = 2k + 1.$$

Let $g = (V_0', V_1', V_2')$ be a Roman dominating function of$\mathcal{G}$, where V_2' dominates V_0'. Then $g(V) =$

$$\sum_{v\in V'} g(v) = \sum_{v\in V_0'} g(v) + \sum_{v\in V_1'} g(v) + \sum_{v\in V_2'} g(v)$$

$= |V_1'| + 2|V_2'|$

Since $V_1 \cup V_2$ is a minimum dominating set of $\mathcal{G}$ (by Theorem 4.1), we have $|V_1| + |V_2| < |V_2'|$

This implies that $|V_2| < |V_2'|$.

So, $g(V) = |V_1'| + 2|V_2'| > |V_1| + 2|V_2| = f(V)$.

That is $f(V)$ is the minimum weight of $\mathcal{G}$, where $f(V_0, V_1, V_2)$ is a Roman dominating function.

Thus $\gamma_R(\mathcal{G}) = 2k + 1$.

Case 2: Suppose $n = 3k + 3$, where $k = 1,2,3 \ldots \ldots$

Now we proceed as in Case1.

Let $V_1 = \{\emptyset\}$,

$V_2 = \{v_3, v_6, v_9, v_{11} \ldots \ldots \ldots \ldots \ldots v_n\}$.

$V_0 = V - \{V_2\} = \{v_1, v_2, v_4, \ldots \ldots \ldots \ldots \ldots v_{n-1}\}$

We observe that V_2 is a dominating set of $\mathcal{G}$ (by Theorem 4.1) and the set V_2 dominates V_0.

Therefore $f = (V_0, V_1, V_2)$ is a Roman dominating function of $\mathcal{G}$. We know that $\gamma(\mathcal{G}) = k + 1$.

So $|V_2| = k + 1, |V_1| = 0, |V_0| = n - k - 1$.

Therefore

$$\sum_{v\in V} f(v) = \sum_{v\in V_0} f(v) + \sum_{v\in V_1} f(v) + \sum_{v\in V_2} f(v)$$

$= 0 + 2\ (k + 1) = 2k+2$

Let $g = (V_0', V_1', V_2')$ be a Roman dominating function of $\mathcal{G}$, where V_2' dominates V_0'. Then

$g(V) =$

$$\sum_{v\in V'} g(v) = \sum_{v\in V_0'} g(v) + \sum_{v\in V_1'} g(v) + \sum_{v\in V_2'} g(v)$$

$= |V_1'| + 2|V_2'|$

Since V_2 is a minimum dominating set of $\mathcal{G}$, we have $|V_2| < |V_2'|$ and $|V_1| \leq |V_1'|$.

So $g(V) = |V_1'| + 2|V_2'| > |V_1| + 2|V_2| = f(V)$.

Therefore $f(V)$ is a minimum weight of Roman dominating function f

Therefore $\gamma_R(\mathcal{G}) = 2k + 2$.

Case 3: Suppose $n = 3k + 4$, where $= 1,2,3 \ldots \ldots$.

Now we proceed as in Case1.

Let $V_1 = \{\emptyset\}$;

$V_2 = \{v_3, v_6, v_9, v_{11} \ldots \ldots \ldots \ldots \ldots v_{n-1}\}$.

$V_0 = V - \{V_2\} = \{v_1, v_2, v_4, \ldots \ldots \ldots \ldots \ldots v_n\}$.

We observe that V_2 is a dominating set of $\mathcal{G}$ (by Theorem 4.1) and the set V_2 dominates V_0.

Therefore $f = (V_0, V_1, V_2)$ is a Roman dominating function of $\mathcal{G}$. We know that $\gamma(\mathcal{G}) = k + 1$.

So $|V_2| = k + 1, |V_1| = 0, |V_0| = n - k - 1$.

Therefore

$$\sum_{v\in V} f(v) = \sum_{v\in V_0} f(v) + \sum_{v\in V_1} f(v) + \sum_{v\in V_2} f(v)$$

$= 0 + 2(k + 1) = 2\ k+2$

If $g = (V_0', V_1', V_2')$ is a Roman dominating function of $\mathcal{G}$, then it follows as in Case 2, that $f(V)$ is a minimum weight of $\mathcal{G}$ for the Roman dominating function $f(V_0, V_1, V_2)$.

Thus $\gamma_R(\mathcal{G}) = 2k + 2$.

Theorem 4.5: Let $\mathcal{G}$ be the interval graph with n vertices, where $1 < n < 5$. Then $\gamma_R(\mathcal{G}) = 2$.

Proof: Let $\mathcal{G}$ be the interval graph with n vertices, where $1 < n < 5$.

Case 1: Suppose $n = 2$.

Let v_1, v_2 be the vertices of $\mathcal{G}$.

Let $V_1 = \{\emptyset\}, V_2 = \{v_2\}, V_0 = V - \{V_2\} = \{v_1\}$

Obviously V_2 is a dominating set of $\mathcal{G}$ and the set V_2 dominates V_0.

Therefore $f = (V_0, V_1, V_2)$ is a Roman dominating function of $\mathcal{G}$.

Therefore

$$\sum_{v\in V} f(v) = \sum_{v\in V_0} f(v) + \sum_{v\in V_1} f(v) + \sum_{v\in V_2} f(v).$$

$= 0 + 0+2= 2$

Thus $\gamma_R(\mathcal{G}) = 2$.

Case 2: Suppose $n = 3$.

Let v_1, v_2, v_3 be the vertices of $\mathcal{G}$.

Let $V_1 = \{\emptyset\}, V_2 = \{v_2\}, V_0 = V - \{V_2\} = \{v_1, v_3\}$.

Here V_2 is a dominating set of $\mathcal{G}$ and the set V_2 dominates V_0. Now we proceed as in Case 1, so that we have $\gamma_R(\mathcal{G}) = 2$.

Case 3: Suppose $n = 4$.

Let v_1, v_2, v_3, v_4 be the vertices of $\mathcal{G}$.

Let $V_1 = \{\emptyset\}, V_2 = \{v_3\}$,

$V_0 = V - \{V_2\} = \{v_1, v_2, v_4\}$.

Here V_2 is a dominating set of $\mathcal{G}$ and the set V_2 dominates V_0. In similar lines to Case 1, we get $\gamma_R(\mathcal{G}) = 2$.

Theorem 4.6: For the Interval graph $\mathcal{G}$, $\gamma(\mathcal{G}) \leq \gamma_R(\mathcal{G}) \leq 2\,\gamma(\mathcal{G})$.

Proof: Let $\mathcal{G}$ be the interval graph. Then by Theorem 4.1, we have $\gamma(\mathcal{G}) = k + 1$.

By Theorem 4.4, we have $\gamma_R(\mathcal{G}) = 2k + 1$ for $n = 3k + 2$ and $\gamma_R(\mathcal{G}) = 2k + 2$ for

$n = 3k + 3, 3k + 4$Where
$k = 1,2,3 \ldots\ldots$Respectively.
Then clearly we have$\gamma(\mathcal{G}) \leq \gamma_R(\mathcal{G}) \leq 2\,\gamma(\mathcal{G})$.
Theorem 4.7: Let $\mathcal{G}$ be the interval graph with n vertices, where $1 < n < 5$ .Then$\gamma_R(\mathcal{G}) = \gamma(\mathcal{G}) + 1$.
Proof: Let $\mathcal{G}$ be the interval graph with n vertices, where$1 < n < 5$.
For$n = 2\,, 3, 4$, by Theorem 4.3 we have $\gamma(\mathcal{G}) = 1$ and by Theorem 4.5 we have $\gamma_R(\mathcal{G}) = 2$.
Therefore $\gamma_R(\mathcal{G}) = 2 = \gamma(\mathcal{G}) + 1$ for$n = 2\,, 3, 4$.
Theorem 4.8: Let $\mathcal{G}$ be the Interval graph of with n vertices. Then $\gamma_R(\mathcal{G}) = \gamma(\mathcal{G}) + k$ for$n = 3k + 2$, where $k = 1,2,3 \ldots\ldots$respectively.
Proof: Let $\mathcal{G}$ be the Interval graph. Then by Theorem 4.1, we have
$\gamma(\mathcal{G}) = 2$ for $n = 5$
= 3 for $n = 8$
= 4 for $n = 11$
And so on.
By Theorem 4.4, we have
$\gamma_R(\mathcal{G}) = 3$ for $n = 5$
= 5 for $n = 8$
= 7 for $n = 11$
And so on.
So, clearly $\gamma_R(\mathcal{G}) = \gamma(\mathcal{G}) + k$ for
$n = 3k + 2$ Where$k = 1,2,3 \ldots\ldots$respectively.
Theorem 4.9: Let $\mathcal{G}$ be the interval graph with n vertices, where n= $3k + 3, 3k + 4$ and
$k = 1,2,3 \ldots\ldots$Respectively. Then $\mathcal{G}$ is a Roman graph.
Proof: Let $\mathcal{G}$ be the interval graph with n vertices, where $n = 3k + 3, 3k + 4$ and
$k = 1,2,3 \ldots\ldots$Respectively.Then by Theorem 4.4, the Roman domination number is

$$\gamma_R(\mathcal{G}) = 2k + 2$$
$$= 2(k + 1)$$
$$= 2\,\gamma(\mathcal{G})$$

Therefore $\mathcal{G}$ is a Roman graph.
Theorem 4.10: Let $\mathcal{G}$ be the interval graph with n vertices. Then $\mathcal{G}$ is a Roman graph if and only if there is a γ_{R-}function $f = (V_0, V_1, V_2)$with$|V_1| = 0$.

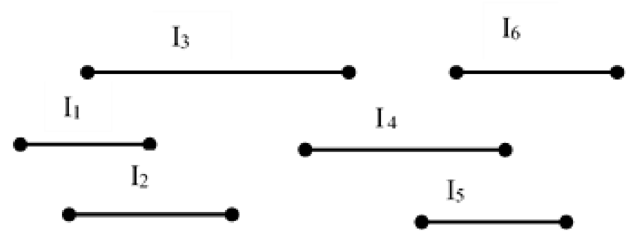
Proof: Let $\mathcal{G}$ be the interval graph with n vertices. Suppose $\mathcal{G}$ is a Roman graph. Let $f = (V_0, V_1, V_2)$ be a γ_{R-}function of$\mathcal{G}$. Then we know that V_2 dominates V_0 and $V_1 \cup V_2$ dominates V. Hence $\gamma(\mathcal{G}) \leq |V_1 \cup V_2|$ = $|V_1| + |V_2| \leq |V_1| + 2|V_2|$=$\gamma_R(\mathcal{G})$. But $\mathcal{G}$ is a Roman graph. So $\gamma_R(\mathcal{G}) = 2\,\gamma(\mathcal{G})$. Then it follows that$|V_1| = 0$, which establishes Case 2, 3 of Theorem 4.4.

Conversely, suppose there is a γ_{R-}function $f = (V_0, V_1, V_2)$ of $\mathcal{G}$ such that$|V_1| = 0$. By the definition of γ_{R-}function, we have $V_1 \cup V_2$ dominates V and since$|V_1| = 0$, it follows that V_2 dominates V. As V_2 is a minimum dominating set of$\mathcal{G}$, we have $\gamma(\mathcal{G})$=$|V_2|$. By the definition of γ_{R-}function we have $\gamma_R(\mathcal{G})$ = $|V_1| + 2|V_2|$ = 0 +2$|V_2|$ = 2$\gamma(\mathcal{G})$.

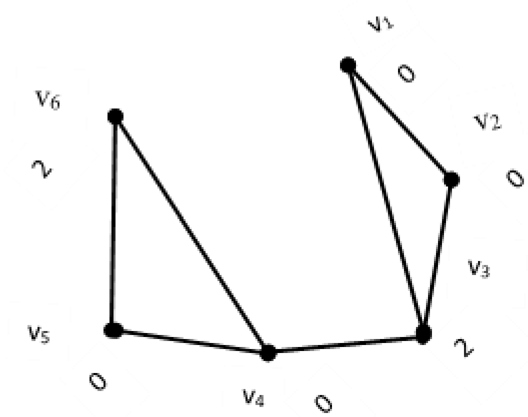
Hence$\mathcal{G}$ is a Roman graph, which also establishes Case 2, 3 of Theorem 4.4.

V. ILLUSTRATIONS

Illustration 1



Interval Family



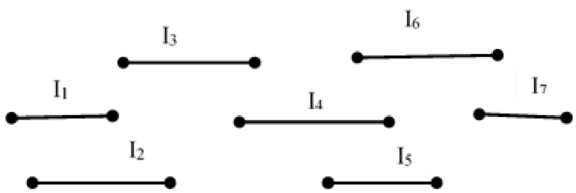
Interval Graph

$D = \{v_3, v_6\}$ and $\gamma(G) = 2$.
$V_1 = \{\emptyset\}$, $V_2 = \{v_3\,, v_6\}$, V_0= V-$\{V_2\}$= $\{v_1, v_2, v_4, v_5\}$

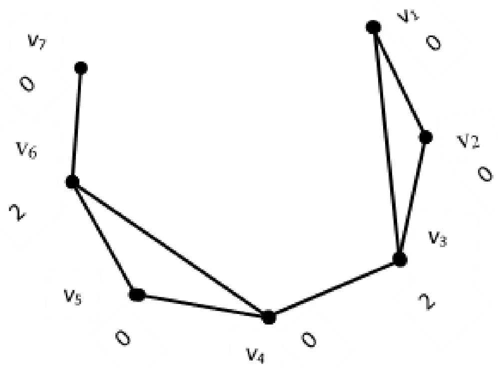
$$\sum_{v \in V} f(v) = |V_1| + 2|V_2| = 0 + 2.2 = 4 = f(V)$$

Therefore $\gamma_R(G) = 4$.

Illustration 2:



Interval Family



Interval Graph

$D = \{v_3, v_6\}$ and $\gamma(G) = 2$.
$V_1 = \{\emptyset\}$, $V_2 = \{v_3 , v_6\}$,
V_0= V-$\{V_2\}$= $\{v_1, v_2, v_4, v_5, v_7\}$

$$\sum_{v \in V} f(v) = |V_1| + 2|V_2| = 0 + 2.2 = 4 = f(V)$$

Therefore $\gamma_R(G) = 4$.

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