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Finite Difference Methods for Two Dimensional Heat Equation

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Abstract- ***In this paper, we first consider the initial boundary value problem for the heat equation. And, the finite difference methods for the heat equation in one space dimension, its consistency and stability are studied. Then, the two dimensional heat equation is solved by using finite difference methods.***

Indexed Terms- ***Consistency, stability, finite difference methods and heat Equation.***

I. INTRODUCTION

We consider the heat equation

$$u_t - \text{div}\,(a(x)\nabla u) = 0, \quad t \in R^+, x \in \Omega,$$

Where u denotes the temperature and a defines the heat conductivity. The generic domain for the heat equation is the space-time cylinder

$$Q_T = (0,T)\times\Omega.$$

In order to obtain a well-posed problem, we must prescribe spatial boundary condition

$$u(t,x) = g(t,x) \;\; \forall\, x \in \partial\Omega, t \in (0,T)$$

and an initial condition

$$u(0,x) = u_0\,(x), x \in \Omega.$$

The equation combined with the initial datum forms an initial boundary value problem.

Note that, the operator Lu = – div (a(x) ∇u) is elliptic, and that in general, the equation with an elliptic differential operator L

$$u_t + Lu = 0$$

Defines a parabolic equation.

We consider the initial boundary value problem (IBVP)

$$u_t + Lu = f(t,x), \quad (t,x) \in Q_T \qquad (1)$$

$$u\,(0, x) = u_0(x), \quad x \in \Omega$$

$$u\,(t, x) = 0, \qquad (t, x) \in \sum_T = (0,T)\times\partial\Omega,$$

Where any inhomogeneous boundary conditions have been eliminated.

If we look for classical solutions of the IBVP, whose derivatives are continuous on $\bar{Q}_T$, the following compatibility condition must also be satisfied

$$u_0(x) = 0 \;\; \forall x \in \partial\Omega.$$

II. FINITE DIFFERENCE METHODS IN ONE SPACE DIMENSION

First, we consider the initial boundary value problem in one space dimension

$$\left.\begin{aligned} u_t - u_{xx} &= f \quad, (t, x) \in (0, T) \times (0, 1) \\ u(0, x) &= u_0(x)\,, x \in (0, 1) \\ u(t, 0) &= u_1(t)\,, t \in (0, T) \\ u(t, 1) &= u_2(t), t \in (0, T). \end{aligned}\right\}(2)$$

We discretise the equation equidistantly with respect to space and time

$$x_i = ih, \quad i = 0, \ldots, N$$
$$t_k = k\tau, \quad k = 0, \ldots, M.$$

We use u_i^k to denote the approximation for $u(t_k, x_i)$ and write $\tilde{f}_i^k = f(t_k, x_i)$.

We require approximations to the first order time derivative and the second order spatial derivative. We use the second difference operator for the spatial variables

$$D^+D^-u_i^k = \frac{1}{h^2}(u_{i-1}^k - 2u_i^k + u_{i+1}^k).$$

If we choose a 2-point discretisation for the time derivative, we obtain a 6-point scheme, which in general has the form

$$\frac{u_i^{k+1} - u_i^k}{\tau} = D^+D^-\,(\sigma u_i^{k+1} + (1-\sigma)u_i^k) + \tilde{f}_i^k$$

With the additional conditions

$u_i^0 = u_0(x_i),\ i = 0, ..., N$

$u_0^k = u_1(t^k),\ k = 0, ..., M$

$u_N^k = u_2(t^k).$

The parameter $\sigma \in [0, 1]$ can be chosen freely.

By defining $\gamma = \frac{\tau}{h^2}$,

We obtain the following special cases:

1. Explicit method ($\sigma = 0$)

$$u_i^{k+1} = (1-2\gamma)u_i^k + \gamma(u_{i-1}^k + u_{i+1}^k) + \tau\tilde{f}_i^k$$

2. Implicit method ($\sigma = 1$)

$$(1+2\gamma)u_i^{k+1} - \gamma(u_{i+1}^{k+1} + u_{i-1}^{k+1}) = u_i^k + \tau\tilde{f}_i^k$$

3. Crank-Nicolson method ($\sigma = \frac{1}{2}$)

$$(\gamma+1)u_i^{k+1} - \frac{\gamma}{2}(u_{i+1}^{k+1} + u_{i-1}^{k+1})$$
$$= (1-\gamma)u_i^k + \frac{\gamma}{2}(u_{i+1}^k + u_{i-1}^k) + \tau f\,(t_k + \frac{\tau}{2}, x_i).$$

For the first method we can proceed iteratively, whereas the last two methods require the solution of a linear system. The coefficient matrices of the systems are always positive definite. Since the matrices are tridiagonal and we can solve such systems with complexity O (N), the complexity is not much greater than that of explicit methods. Later, we will see that the heat equation is stiff, and that therefore implicit methods are advantageous. The methods differ especially in their stability and consistency properties.

2.1 Theorem

The consistency error satisfies

1. O ($h^2 + \tau$) for arbitrary σ and $\tilde{f}_i^k = f(t_k, x_i)$ where $u \in C^{4,2}(\bar{Q}_T)$.

2. O ($h^2 + \tau^2$) for $\sigma = \frac{1}{2}$ and $\tilde{f}_i^k = f(t_k + \frac{\tau}{2}, x_i)$ where $u \in C^{4,3}(\bar{Q}_T)$.

Proof

For the first part, we use the Taylor expansions

$$u(t+\tau, x) = u(t, x) + \tau u_t(t, x) + \frac{\tau^2}{2!}u_{tt}(t,x) + \frac{\tau^3}{3!}u_{ttt}(t,x) + \frac{\tau^4}{4!}u_{tttt}(t,x)$$

$$\frac{u(t+\tau,x) - u(t,x)}{\tau} = u_t(t,x) + \frac{1}{2}\tau u_{tt}(t,x) + \frac{\tau^2}{3!}u_{ttt}(t,x) + \frac{\tau^3}{4!}u_{tttt}(t,x) = u_t + O(\tau)$$

$$u(t, x+h) = u(t,x) + hu_x(t,x) + \frac{h^2}{2!}u_{xx}(t,x) + \frac{h^3}{3!}u_{xxx}(t,x) + \frac{h^4}{4!}u_{xxxx}(t,x),$$

$$u(t, x-h) = u(t,x) - hu_x(t,x) + \frac{h^2}{2!}u_{xx}(t,x) - \frac{h^3}{3!}u_{xxx}(t,x) + \frac{h^4}{4!}u_{xxxx}(t,x),$$

$$\frac{u(t,x-h) - 2u(t,x) + u(t,x+h)}{h^2} = u_{xx}(t,x) + \frac{2h^2}{4!}u_{xxxx}(t,x),$$

$$\frac{u(t,x-h) - 2u(t,x) + u(t,x+h)}{h^2} = u_{xx}(t,x) + O(h^2).$$

Moreover, we have $\tilde{f}_i^k = f(t_k, x_i).$

Therefore, consistency error satisfies

$$e = (u_t - u_{xx} - f) + O(\tau) + O(h^2),$$

$$e = O(\tau) + O(h^2) = O(\tau + h^2).$$

For the second part, we use the Taylor expansions

$$\frac{u(t+\tau,x) - u(t,x)}{\tau} = u_t(t,x) + \frac{\tau}{2!}u_{tt}(t,x) + \frac{\tau^2}{3!}u_{ttt}(t,x) + \frac{\tau^3}{4!}u_{tttt}(t,x),$$

$$\frac{u(t+\tau,x) - u(t,x)}{\tau} = u_t + \frac{\tau}{2}u_{tt} + O(\tau^2)$$

and

$$\frac{u(t,x-h) - 2u(t,x) + u(t,x+h)}{h^2} = u_{xx}(t,x) + \frac{2h^2}{4!}u_{xxxx}(t,x),$$

$$\frac{u(t+\tau,x-h)-2u(t+\tau,x)+u(t+\tau,x+h)}{h^2}$$

$$=u_{xx}(t,x)+\frac{2h^2}{4!}u_{xxxx}(t,x)$$

$$+\tau u_{xxt}(t,x)+\frac{\tau^2}{2!}u_{xxtt}(t,x),$$

$$\frac{1}{2}\left[\frac{u(t+\tau,x-h)-2u(t+\tau,x)+u(t+\tau,x+h)}{h^2}+\frac{u(t,x-h)-2u(t,x)+u(t,x+h)}{h^2}\right]$$

$$=\frac{1}{2}\left[2u_{xx}(t,x)+\tau u_{xxt}(t,x)+\frac{h^2}{6}u_{xxxx}(t,x)+\frac{\tau^2}{2!}u_{xxtt}(t,x)\right],$$

$$\frac{1}{2}\left[\frac{u(t+\tau,x-h)-2u(t+\tau,x)+u(t+\tau,x+h)}{h^2}+\frac{u(t,x-h)-2u(t,x)+u(t,x+h)}{h^2}\right]$$

$$=\frac{1}{2}\left[2u_{xx}(t,x)+\tau u_{xxt}(t,x)+O(\tau^2+h^2)\right].$$

Moreover, we have

$$f\left(t+\frac{\tau}{2},x\right)=f(t,x)+\frac{\tau}{2}f_t(t,x)+O(\tau^2).$$

Therefore, consistency error satisfies

$$e=(u_t-u_{xx}-f)+\frac{1}{2}\tau(u_{tt}-u_{xxt}-f_t)+O(\tau^2+h^2)$$

$$e=O(h^2+\tau^2). \quad \square$$

Definition 2.2. Let f = 0. The method is called stable with respect to the norm $\|.\|$, if

$$\|u(t_k)\|\le C\|u(0)\|.$$

Now define $\|.\|=\|.\|_{L^2}$ the L^2-norm. We want to investigate stability in this norm using Von Neumann's approach, which is based on Fourier analysis. The discrimination of the homogeneous differential equation with homogeneous boundary conditions is then given by

$$\frac{u_i^{k+1}-u_i^k}{\tau}=D^+D^-(\sigma u_i^{k+1}+(1-\sigma)u_i^k)$$

$$u_i^0=u_0(x_i),\ i=0,\dots,N$$

$$u_0^k=u_N^k=0,\ k=0,\dots,M.$$

Analogously to the continuous case, a solution can be found by using a separation of variables method as $u_i^k=v_iw^k$.

Substituting in the equation, we have

$$\frac{w^{k+1}-w^k}{\tau}v_i=D^+D^-v_i(\sigma w^{k+1}+(1-\sigma)w^k)$$

Respectively,

$$\frac{w^{k+1}-w^k}{\tau(\sigma w^{k+1}+(1-\sigma)w^k)}=\frac{D^+D^-v_i}{v_i}=\text{const}=-\lambda.$$

Thus the v_i solves the discrete eigenvalue problem

$$D^+D^-v_i+\lambda v_i=0$$
$$v_0=v_N=0$$

and moreover, we have

$$w^{k+1}-w^k=-\lambda_k\tau(\sigma w^{k+1}+(1-\sigma)w^k)$$

$$w^{k+1}+\lambda_k\tau\sigma w^{k+1} = w^k-\lambda_k\tau(1-\sigma)w^k$$

$$w^{k+1}=\left(\frac{(1-(1-\sigma)\tau\lambda_k)}{1+\tau\sigma\lambda_k}\right)w^k$$

$$w^{k+1}=q_kw^k,$$

where $q_k=\dfrac{1-(1-\sigma)\tau\lambda_k}{1+\sigma\tau\lambda_k}$.

We now consider the discretization in L^2-norm of

$$u^k=\sum_i w^kv_i,$$

then we have

$$\|u^k\|^2=h\sum_i(w^k)^2v_i^2$$
$$\le hq_{k-1}(w^{k-1})^2\sum_i v_i^2$$
$$\le hq_{k-1}q_{k-2}(w^{k-2})^2\sum_i v_i^2$$
$$\le q_{k-1}\dots q_0h\sum_i(w^0)^2v_i^2$$
$$=q_{k-1}\cdots q_0\|u_0\|^2.$$

For the right-hand side to be bounded as $k\to\infty$, we require that

$$|q_k|\le 1 \text{ For all eigenvalues } \lambda_k.$$

Then, we have stability in the discrete L^2-norm.

In particular, we have

- Explicit Method ($\sigma = 0$): $q_k = 1-\tau\lambda_k$,
- Implicit Method ($\sigma = 0$) : $q_k=\dfrac{1}{1+\tau\lambda_k}$,
- Crank-Nicolson Method $(\sigma=\frac{1}{2}): q_k=\dfrac{1-\frac{1}{2}\tau\lambda_k}{1+\frac{1}{2}\tau\lambda_k}$.

2.3 THE METHOD OF VERTICAL LINES

In the following, let $\Omega\subset\mathbb{R}^n$ and we consider the initial boundary value problem,

$$u_t + Lu = f(t,x), \qquad (t,x)\in Q_T$$
$$u(0,x) = u_0(x), \qquad x\in\Omega$$

$$u(t, x) = 0, (t, x) \in \sum_T = (0,T)\times\partial\Omega$$

Where the differential operator L is uniformly elliptic and $f \in L^2(Q_T)$.

The idea of the method of lines is to consider the spatial and the time discretisation subsequently, i.e., one after another. In the method of vertical lines the spatial coordinates are discretised first. In a system of ordinary differential equations, we can discretise using known methods.

2. 4 Example

We consider the initial boundary value problem for one dimensional heat equation,

$$u_t - u_{xx} = 0,$$
$$u(t,0) = u(t,1) = 0$$
$$u(0,x) = u_0(x).$$

Then,

$$\frac{d}{dt}u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \qquad i = 1,\dots,N-1$$

$$u_i(0) = u_0(x_i), \text{ respectively}$$

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ M \\ u_{N-1} \end{pmatrix} = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & L & 0 \\ 1 & -2 & 1 & & M \\ & & & & \\ 0 & O & O & O & 0 \\ M & & 1 & -2 & 1 \\ 0 & L & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ M \\ u_{N-1} \end{pmatrix},$$

$$u_i(0) = u_0(x_i).$$

In order to apply this approach as easily as possible to multi-dimensional problems.

III. FINITE DIFFERENCE METHODS IN TWO SPACE DIMENSIONS

Now we consider the heat equation in two spatial dimensions

$$u_t = u_{xx} + u_{yy}$$

With initial condition u(x, y, 0) = u_0(x, y) and boundary conditions onΩ. We can discretise the right hand side of this equation by using the five-point difference stencil on the Laplace operator,

$$\Delta_h u(x_i, y_i) = \frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}.$$

If we now apply the trapezium rule for the time discretisation, we obtain the two-dimensional version of the Crank-Nicolson method

$$u_{ij}^{k+1} = u_{ij}^k + \frac{\tau}{2}(\Delta_h u_{ij}^k + \Delta_h u_{ij}^{k+1}).$$

Since this method is implicit, we must again solve a linear system of equations, the structure of which is exactly the same as in the stationary case. Due to the fact that the system has a sparse, very high-dimensional matrix, we should solve using an iterative method. This is all the more relevant, since we must solve such a system on every time level, i.e., we must be prepared to solve the system several hundred times.

Now we write the above formula as

$$\left(I - \frac{\tau}{2}\Delta_h\right)u_{ij}^{k+1} = \left(I + \frac{\tau}{2}\Delta_h\right)u_{ij}^k.$$

Although the matrix has the same structure with respect to the zero elements, we must still take care when scalingτ. We define

$$A = I - \frac{\tau}{2}\Delta_h,$$

And calculates its eigenvalues

$$\lambda_{p_1,p_2} = 1 - \frac{\tau}{h^2}((\cos(p_1\pi h) - 1) + (\cos(p_2\pi h) - 1)),$$

Where p_1, p_2 denote the eigenvalues of Δ_h. the largest eigenvalue has the order of magnitude $1 + O(\frac{\tau}{h^2})$, whereas the smallest has 1+ O (τ). That means, the condition number of the matrix is 1 + $O(\frac{\tau}{h^2})$. as a comparison: the discrete Laplace operator has the condition $O(\frac{1}{h^2})$. Therefore we can expect iterative methods to converge much quicker by the smaller condition number. And apart from that, we have good initial value for the iteration, i.e., the solution from the previous time level.

In order to obtain better starting approximations, we can naturally extrapolate forwards and use,

For example, $u_{ij}^{[0]} = 2u_{ij}^k - u_{ij}^{k-1}$.

Or, we can perform an Euler step

$$u_{ij}^{[0]} = \left(I + \frac{\tau}{2}\Delta_h\right)u_{ij}^k.$$

An alternative to solving a fully coupled linear system is to replace the time step by a sequence of time steps, calculating only one spatial dimension in each one. Then, we only have to solve a tridiagonal system. The Locally One-Dimensional (LOD) method can be expressed as

$$u_{ij}^{*} = u_{ij}^{k} + \frac{\tau}{2}(D_x^2 u_{ij}^{k} + D_x^2 u_{ij}^{*})$$

$$u_{ij}^{k+1} = u_{ij}^{*} + \frac{\tau}{2}(D_y^2 u_{ij}^{*} + D_y^2 u_{ij}^{k+1})$$

or, in matrix form

$$\left(I - \frac{\tau}{2} D_x^2\right) u^{*} = \left(I + \frac{\tau}{2} D_x^2\right) u^{k}$$

$$\left(I - \frac{\tau}{2} D_y^2\right) u^{k+1} = \left(I + \frac{\tau}{2} D_y^2\right) u^{*}.$$

Here we initially solve with the Crank-Nicolson method in the x direction and obtain u^*, which enables us to solve with this result in the y direction. First, we only consider the diffusion in one coordinate direction and then in the other. If the time step is very small, we can expect (and sometimes even prove), that the solution shows the same behaviour.

A modification of the LOD method results from a further splitting method, for which we must also only solve a tridiagonal system, since in every step, we only have to solve in one direction. However, if we exchange the directions, using the so called Alternative Direction Implicit (ADI) method, we have

$$u_{ij}^{*} = u_{ij}^{k} + \frac{\tau}{2}(D_y^2 u_{ij}^{k} + D_x^2 u_{ij}^{*})$$

$$u_{ij}^{k+1} = u_{ij}^{*} + \frac{\tau}{2}(D_x^2 u_{ij}^{*} + D_y^2 u_{ij}^{k+1})$$

or in matrix form

$$\left(I - \frac{\tau}{2} D_x^2\right) u^{*} = \left(I + \frac{\tau}{2} D_y^2\right) u^{k}$$

$$\left(I - \frac{\tau}{2} D_y^2\right) u^{k+1} = \left(I + \frac{\tau}{2} D_x^2\right) u^{*}.$$

In both steps we have the diffusion in both directions where one component is dealt with explicitly and the other implicitly.

IV. CONCLUSION

This paper has presented the heat equation in one space dimension and two space dimensions. Next, its consistency is calculated by using Taylor's expansions. Then, we investigate their stability properties for one dimensional heat equation. Next, we study two dimensional heat equation by finite difference methods.

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