No Separable Components, Chordality and 2-Factors in Tough Graphs

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Abstract- In this paper we mention non separable components of longest cycles. And then we establish chordality and 2-factor in tough graph. A graph G is chordal if it contains no chordless cycle of length at least four and is k-chordal if a longest chordless cycle in G has length at most k. Finally the result reveals that all 3/2-tough 5-chordal graph G with a 2-factor are obtained.

Indexed Terms- non separable components, 2-factor, induced subgraph, toughness, maximum degree, minimum degree, longest cycle, chordal graph, Tutte pair.

I. INTRODUCTION

A graph is finite if its vertex set and edge set are finite. A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A complete graph G is a simple graph in which every pair of vertices is adjacent. A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X, Y) is called a bipartition of the graph. A complete bipartite graph is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y : if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$.

Suppose that V' is a nonempty subset of V. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G induced by V' and is denoted by $G[V']$; we say that $G[V']$ is an induced subgraph of G. Now suppose that E' is a nonempty subset of E. The subgraph of G whose vertex set is the set of ends of edges in E' and whose edge set is E', is called the subgraph of G induced by E' and is denoted by $G[E']$; $G[E']$ is an edge - induced subgraph of G. A vertex-cut in a graph G is a set U of vertices of G such that $G - U$ is disconnected. A complete graph has no vertex-cut.

The vertex-connectivity or simply the connectivity $\kappa(G)$ of a graph G is the minimum cardinality of a vertex-cut of G if G is not complete , and $\kappa(G) = n - 1$ if $G = K_n$ for some positive integer n . Hence $\kappa(G)$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. If G is either trivial or disconnected, $\kappa(G) = 0$. G is said to be k- connected if $\kappa(G) \geq k$. All non-trivial connected graphs are 1 -connected. If G is non complete graph and t is a nonnegative real number such that $t \leq \frac{|S|}{\omega(G-S)}$ for every vertex - cut S of G , then G is defined to be t -tough. If G is a t - tough graph and s is a nonnegative real number such that s < t, then G is also s-tough. The maximum real number t for which a graph G is a t-tough is called the toughness of G and is denoted by t(G) . A connected graph with at least one cut vertex is called a separable graph, otherwise it is no separable.). The degree (or valence) of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges.

We denote by $\delta(G)$ and μ_G the minimum and maximum degrees, respectively of vertices of G. The set of neighbours of a subgraph H of G , denoted N(H), is the set of vertices in $V(G) - V(H)$ adjacent to at least one vertex of H ; $d(H) = |N(H)|$ is the degree of a subgraph H of G.

II. NONSEPARABLE COMPONENTS

We still have to investigate non separable components of longest cycles. For an induced subgraph H of G, we set

$X(H)=\{x\in N_{G-H}(H):|N_H(x,x')|\geq 2 \text{ for each } x'\in N_{G-H}(H)-\{x\}\}$

And $Y(H)=\{y\in V(H): N_{G-H-X(H)}(y)\neq\varnothing\}$.

Notice that, in fact, $|N_{G-H-X(H)}(y)|\geq 2$ for all $y \in Y(H)$.

If $V(H)=Y(H)$, set $\mu_H=\max_{v\in V(H)} d(v)$. Otherwise, let $\mu_H=|X(H)\cup Y(H)|$.

If $N_{G-H}(H)\neq V(G-H)$ and $V(H)\neq Y(H)$, then $X(H)\cup Y(H)$ is a vertex-cut of G. Anyway,
$\mu_H \geq \kappa_G \geq 2t$, where κ_G denotes the connectivity of G.

2.1 Lemma

Let C be a cycle in a graph G such that $h=c(G)-|C|$, and let H be a component of G – C. Further, let x_1, x_2 be distinct vertices on C and let $v_1\in N_H(x_1)$, $v_2\in N_H(x_2)$. Let P be a longest (v_1,v_2)-path in H, and let $Q=Q[z_1,z_2]$ be a C-arc from $C(x_1,x_2]$ to $C(x_2,x_1]$ such that $Q\cap P=\varnothing$. Then

(i) $|C(x_1,x_2)|\geq D_H(v_1,v_2)+1-h$ And

(ii)

$|C(x_1,z_1)|+|C(x_2,z_2)|\geq D_H(v_1,v_2)+1+(|Q|-2)-h.$

2.2 Lemma

Let C be a longest cycle in a 2-connected graph G, and let H be a 2-connected component of G – C such that $|C| < (t+1)(\mu_H+D(H))+t$. Then $D < 2t$ and

$|C|\geq \mu_H(D(H)+2)+\min(D(H)+2,\ t+1)$. (1)

Proof

We fix a cyclic orientation on C and abbreviate X = X (H), Y = Y (H), and D = D (H). Note that $D \geq 2$, since H is 2-connected.

If $D \geq 2t$ and (1), we obtain a contradiction to the hypothesis of Lemma 2.2, since

$$\mu_H(D+2)=\frac{\mu_H+2}{2}D+D\frac{\mu_H-2}{2}+2\mu_H$$

$$\mu_H(D+2)\geq (t+1)D+t(\mu_H-2)+2\mu_H$$

$$\geq (t+1)(D+\mu_H).$$

Therefore, it suffices to show (1).

For $x\in N_C(H)$, let x^* denote the first vertex on C (x, x] such that $x^*\in N_C(H)$. Let X denote the set of all $x \in N_C(H)$ such that $|N_H(x,x^*)|\geq 2$ and label

$X=\{x_1,\ldots,x_m\}$, according to the given orientation. Then $|C(x_i,x_i^*)|\geq D+1$ Lemma 2.1 therefore

$|C| = m(D+2)+R_1$, where $R1\geq 2|N_{C-X}(H)|$. (2)

If $m>\mu_H$, then (2) immediately yields (1). For the rest of the proof, let $m=\mu_H$. Then, for each $y \in Y$, there exists a unique vertex $\hat{y}\in N_{C-X}(y)\cap\hat{X}$ and, therefore, $|N_H(x_j,x_k)|\geq 2$ for any distinct $x_j,x_k\in X$.

Consider a C-arc $Q=Q[z_j,z_k]$ between distinct segments $C(x_j,x_{j+1})$ and $C(x_k,x_{k+1})$ such that $Q\cap V(H)=\varnothing$. By Lemma 2.1, we have

$|C(x_j,z_j)|+|C(x_k,z_k)|\geq D+1.$ (3)

Since $|N_H(x_{j+1},x_{k+1})|\geq 2$, we are allowed to use the same argument with the orientation of C reversed. Therefore

$|C(z_j,x_{j+1})|+|C(z_k,x_{k+1})|\geq D+1$ and, by (3)

$|C(x_j,x_{j+1})|+|C(x_k,x_{k+1})|\geq 2D+4.$ (4)

If $R_1 \leq 1$, then clearly $N_C(H)=X$ and, by (4), there is no C-arc Q between distinct segments of the for $C(x_i,x_{i+1})$. In this case, $t\leq\frac{m}{m+1}$ and (2) yields a contradiction, since

$m(D+2)\geq 2m+2D=\frac{2m+1}{m+1}(m+D)+\frac{1}{m+1}(m+D)>(t+1)(\mu_H+D)+1.$

For the rest of the proof, let us assume $2 \leq R_1 < t+1$.

For h = 0, 1, set $X_h = \{x_j \in X : |C(x_j, x_{j+1})| = D+1+h\}$, and let $X_2 = \hat{X} - (X_0 \cup X_1)$.

For $x_i \in X_0$, set $w_i = x_{i+1}$. If $X_1 \neq \varnothing$ pick $x_s \in X_1$, set ε = 1 and $w_S = x_{s+1}^{-}$.

If $X_1 = \varnothing$, set ε=0. Further, let $w_i = x_i^{++}$ for $x_i \in \hat{X} - (X_0 \cup \{x_s\})$.

It readily follows from (3) and (4) that there is no C-arc Q between distinct segments of the form $C(x_i, w_i)$. Therefore, G − $V\left(\bigcup_{i=1}^{m} C[w_i, x_{i+1}]\right)$ has at least m + 1 components and, consequently,

t (m + 1) ≤ $|C| - \left|V\left(\bigcup_{i=1}^{m} C(x_i, w_i)\right)\right|$.

Hence $|C|$ ≥ (t + 1) m + t + $(|X_0| + \varepsilon)$ D. (5)

As $|X_1| + 2|X_2|$ ≤ R_1 < t + 1 ≤ $\frac{m+2}{2}$. We have $|X_1| + 2|X_2|$ ≤ $\frac{m+1}{2}$, by (5) and the hypothesis of Lemma 2.2 also $|X_0| + \varepsilon$ < t + 1, and, hence,

$|X_0| + \varepsilon$ ≤ $\frac{m+1}{2}$. Now

$|X_0| + |X_1| + 2|X_2| + \varepsilon$ ≤ m + 1 and $|X_2| + \varepsilon$ ≤ 1.

On the other hand, $|X_2| + \varepsilon$ ≥ 1, since R_1 > 0.

Consequently, $|X_2| + \varepsilon$ = 1 and

$|X_0| + \varepsilon$ = $|X_1| + 2|X_2|$ = $\frac{m+1}{2}$. (6)

If $X_2 = \varnothing$ then Y =∅, and G − X = G − X has at least $|X_0| + 2$ components. In this event,

$1 < t \leq \frac{m}{|X_0| + 2} = \frac{2|X_0| + 1}{|X_0| + 2} < 2$,

But $|X_0| + \varepsilon$ = $|X_0| + 1$ < t + 1 and, therefore, $|X_0|$ = 1, A contradiction.

Hence, in fact, $|X_2|$ = 1 and ε = 0. Consequently, $X_1 = \varnothing$ and, by (6), m = 3 ≥ 2t. Since D ≥ 2,
We obtain from (2)

$|C| = \frac{5}{2}(D + 3) - \frac{5}{2} + \frac{1}{2}(D + 2) + R_1 \geq (t + 1)(m + D) + \frac{3}{2}$. This contradiction completes the proof of Lemma 2.2.

III. CHORDALITY AND 2-FACTORS IN TOUGH GRAPH

G is chordal if it contains no chordless cycle of length at least four and is k-chordal if a longest chordless cycle in G has length at most k.

Let G be a graph. If A and B are subsets of V or subgraphs of G, and v ∈ V, we use e (v, B) to denote the number of edges joining v to a vertex of B, and e (A, B) to denote $\sum_{v \in A} e(v, B)$. For disjoint subsets A, B of V (G) let odd (A, B) denote the number of components H of G − (A ∪ B) with e (H, B) odd, and let

ϑ (A, B) = $2|A| + \sum_{y \in B} d_{G-A}(y) - 2|B| - odd(A, B)$.

A Tutte pair for a graph G is a pair (A, B) of disjoint subsets of V (G) with ϑ (A, B) ≤ – 2.

We define a Tutte pair (A, B) to be minimal if ϑ (A, B′) ≥ 0 for any proper subset B′ ⊆ B.

We also define a Tutte pair (A, B) to be a strong Tutte pair if B is an independent set.

3.1 Lemma
Let v be a simplicial vertex in a non-complete graph G. Then t (G – v) ≥ t (G).
Proof

First denote G – v by G_v. Note that if G_v is complete, then t (G_v) = $\frac{|V(G_v)| - 1}{2} = \frac{|V(G)| - 2}{2}$ ≥ t (G).

Suppose t (G_v) < t (G). Then there exists $X \subseteq V(G_v)$ such that $\omega(G_v - X) \geq 2$ and $\frac{|X|}{\omega(G_v - X)}$ < t (G). However $\omega(G - X) \geq \omega(G_v - X) \geq 2$, since the neighbours of v in G induced a complete subgraph. But this gives $\frac{|X|}{\omega(G-X)} \leq \frac{|X|}{\omega(G_v - X)}$ < t (G), a contradiction.

3.2 Theorem
Let G be any graph. Then
(i) For any disjoint set A, B $\subseteq$ V (G), ϑ (A, B) is even;
(ii) The graph G does not contain a 2-factor if and only if ϑ (A, B) ≤ -2 for some disjoint pair of sets A, B $\subseteq$ V (G).

3.3 Lemma
Let G be a graph having no 2-factor. If (A, B) is a minimal Tutte pair for G, then B is an independent set.

3.4 Theorem
Let G be a $\frac{3}{2}$-tough 5-chordal graph. Then G has a 2-factor. Proof Let G be a $\frac{3}{2}$-tough 5-chordal graph having no 2-factor and (A, B) be a strong Tutte pair for G, existing by Lemma 3.3 Thus ϑ (A, B) ≤ -2. Let $C = V(G) - (A \cup B)$. Since B is an independent set of vertices, $\sum_{y \in B} d_{G-A}(y)$ = e (B, C). Hence by Theorem 3.2, $2|A|$ + e(B, C) $\leq 2|B|$ + odd(A, B) – 2. (7)

Among all possible choices, we choose G and the strong Tutte pair (A, B) as follows:
(i) $|V(G)|$ is minimal;
(ii) $|E(G)|$ is maximal, subject to (i);
(iii) $|B|$ is minimal, subject to (i) and (ii);
(iv) $|A|$ is maximal, subject to (i), (ii) and (iii).

We now show that G has properties (a)-(g) below.
(a) For any x $\in$ B and any component H of [C], e(x, H) ≤ 1.
Proof of (a)

Let x $\in$ B with $d_{G-A}(x)$ = k and let $C_1, C_2, \ldots C_j$ denote the components of [C] to which x is adjacent. If j $\leq$ k – 1, delete x from B and add x to C (thus redefining B and C). Since odd (A, B) has decreased by at most j $\leq$ k – 1, it is easy to check that ϑ (A, B) has increased by at most 1. Thus we still have ϑ (A, B) ≤ -2 by Theorem 3.2 (i) and we contradict (iii).

(b) The vertices of A are complete.
Proof of (b)

If not, form a new graph G′ by adding the edges required to make the vertices of a complete. Clearly G′ is still $\frac{3}{2}$-tough and (A, B) is still a strong Tutte pair for G′. Obviously, no chordless cycle of G′ can contain a vertex of A. Since G is 5-chordal, it follows that G′ is also 5-chordal. Thus we contradict (ii).

(c) For any y $\in$ C, e(y, B) ≥ 1.
Proof of (c)

Suppose that e(y, B) ≥ 2 for some y $\in$ C. Delete y from C and add y to A (thus redefining A and C). It is easy to check that (A, B) remains a strong Tutte pair. Thus we contradict (iv).

(d) Each component of [C] is a complete graph.
Proof of (d)
If not, form a new graph G′ by adding the edges required to make each component $C_1, C_2, \ldots, C_s$ of [C] a complete graph. Clearly, G′ is still $\frac{3}{2}$-tough and (A, B) is still a strong Tutte pair for G′. Assuming G′ is not 5-chordal, let C^* be a shortest chordless cycle in G′ of length at least 6. Clearly C^* cannot contain a vertex of A, nor can it have more than two vertices from any component of [C]. Since B is independent, C^* is of the form C^*: $b_1 T_1' b_2 T_2' \ldots b_k T_k' b_1$,
Where, $1 \leq i \leq k$, each T_i' represents an edge $t_i^1 t_i^2$ of a component C_i in G′.

Form the cycle C^{**} in G by taking C^* and substituting T_i for T_i' ($1 \leq i \leq k$), where T_i is a shortest (t_i^1, t_i^2)-path in C_i in G. The graph G is 5-chordal so C^{**} has a chord. Since any chord of C^{**} must join a vertex of B and a vertex of C and C^* is a chordless cycle in G′, we may assume, without loss of generality, that there exists a chord b_1u of C^{**} such that

-u is an internal vertex of some T_i, say of T_m, and

-the cycle $b_1 T_1 b_2 T_2 \ldots b_m U b_1$, where U is the (t_m^1, u)-subpath of T_m, is chordless.

By (a) we have 1<m< k. But then $b_1 T_1' b_2 T_2' \ldots b_m t_m^1 u b_1$ is a chordless cycle in G′ of length at least 6 which is shorter than C^*, contradicting the choice of C^*. Thus G′ is 5-chordal and we contradict

(ii).

(e) For any y ∈ C, e(y, B) = 1 (and thus e (B, C) = $|C|$).

Proof of (e)

Suppose now that C contains a vertex y with e(y, B) = 0. It follows from (b) and (d) that v is simplicial. Hence by Lemma 3.1, t (G – y) ≥ t (G). Furthermore, (A, B) is still a strong Tutte pair for the

5-chordal graph G – y. Hence, by (i), the graph G – y contradicts the choice of G.

(f) $|B| \geq 2$.

Proof of (f)

We saw earlier that $|B| > |A| \geq 0$, and so $|B| \geq 1$. Suppose B = {x}. Since (A, B) is a Tutte pair with $|B|$ = 1 and

$|A|$ = 0, we have e (B, C) ≤ odd (A, B) by (7).

If e (B, C) ≥ 2,

Then ω(G – B) ≥ odd (A, B) ≥ e (B, C) ≥ 2 > $|B|$, and G is not 1-tough. If e (B, C) = 1, then G is not 1-tough either. Hence $|B| \geq 2$.

(g) Odd (A, B) = ω([C]).

Proof of (g)

Suppose there exists a component C_i in [C] with e (C_i, B) = $|C_i|$, an even integer. Let y be any vertex in C_i. Add y to A, thus redefining A and C, it is easy to see that (A, B) is still a strong Tutte pair for G. Thus we contradict (iv).

Hence G and its minimal Tutte pair (A, B) has properties (a)-(g). Set s = ω([C]) = odd (A, B).

Consider the components $C_1, C_2 \ldots C_s$ of [C] and let $y_j \in V(C_j)$. Define $X = A \cup C - \{y_1 \ldots y_s\}$. Since B is independent and e (y_i, B) = 1 for 1 ≤ i ≤ s, we have ω(G – X) = $|B| \geq 2$. For convenience let a = $|A|$, b = $|B|$ and c = $|C|$. Using properties (e), (g) and inequality (7), we have

$$\frac{3}{2} \leq \frac{|X|}{\omega(G-X)} = \frac{a+c-s}{b}$$

$$= \frac{a+e(B,C)-odd(A,B)}{b} \leq \frac{2b-a-2}{b}.$$

Hence b ≥ 2a + 4. (8)

Claim. b ≥ c – s + 1.

Once the claim is established, it follows that

$$\frac{3}{2} \leq \frac{|X|}{\omega(G-X)} = \frac{a+c-s}{b} \leq \frac{a+b-1}{b}.$$

Thus b ≤ 2a – 2.

(9)

The fact that (8) and (9) are contradictory completes the argument.

Proof of Claim

Form a bipartite graph F from G by deleting A and contracting each component of [C] into a single vertex. By (a), F has no multiple edges. The key observation is that since G is 5-chordal, F is a forest. Otherwise, let C_F be a shortest cycle in F. Then C_F is of the form

$C_F: b_1 T_1 b_2 T_2 \ldots b_p T_p b_1$,

Where each T_i, 1 ≤ i ≤ p, represents the contracted component C_i. By (d) and (e), it follows that the 2 edges incident with each T_i in C_F correspond to edges $b_i t_i^1$, $b_{i+1} t_i^2$, where $t_i^1 t_i^2$ in an edge in C_i. If follows that G has a chordless cycle of length at least 6, a contradiction. Hence

$$\sum_{v \in C} d_F(v) = c = |E(F)| \leq |V(F)| - 1 = b + s - 1.$$

Thus b + s – 1 ≥ c and the claim is established.

IV. CONCLUSION

We conclude that deal with toughness relate to cycles structure are expressed. It is shown that 3/2-tough 5-chordal graph G has a 2-factor.

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