© JUL 2019 | IRE Journals | Volume 3 Issue 1 | ISSN: 2456-8880

Convergence for Non-Stationary Advection- Diffusion Equation

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Abstract- In this paper, we first consider parabolic advection-diffusion problem. And, we study a characteristic Galerkin method for non- stationary advection diffusion equation. Then, we prove the stability and convergency of these method.

Indexed Terms- Advection- Diffusion Equation, convergency, characteristic Galerkin method.

I. INTRODUCTION

We assume that Ω is a bounded domain in $R^{n,}$ n=2, 3….with Lipschitz boundary and consider the parabolic initial boundary value problem: For each $t \in [0,T]$, we can find u (t) such that

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + Lu = f \quad &\text{in} \quad Q_T = (0,T)\times\Omega \\ u = 0 \quad &\text{on} \quad \sum_T = (0,T)\times\partial\Omega, \\ u = u_0 \text{ on } \Omega, &\quad \text{for } t=0, \end{aligned}\right\} (1)$$

Where L is the second-order elliptic operator

$$Lw = -\varepsilon\Delta w + \sum_{i=1}^{n} D_i(b_i w) + a_0 w.(2)$$

We consider the case in which $\varepsilon = \|b\|_{L^\infty(\Omega)}$.

The "simplified" form considered in (2) is a perfect alias of the most general situation in which L is given by $Lz = -\sum_{i,j=1}^{n} D_i(a_{ij}D_j z) + \sum_{i=1}^{n} D_i(b_i z) + a_0 z,$ whenever the diffusion coefficients a_{ij} are smaller than the adventive ones $b_i, i, j = 1,...,n.$ without loss of generality, we suppose that b is normalized to $\|b\|_{L^\infty(\Omega)} = 1.$

We assume that there exist two positive constants μ_0 and μ_1 such that

$$0 < \mu_0 \le \mu(x) = \frac{1}{2}\text{div } b(x) + a_0(x) \le \mu_1$$

For almost every $x \in \Omega$.

The parabolic advection-diffusion problem (1) can be reformulated in a weak form as follows:

Given $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$, find $u \in L^2(0,T;V) \cap C^0([0,T];L^2(\Omega))$ such that

$$\left.\begin{aligned} \frac{d}{dt}\big(u(t),v\big) + a\big(u(t),v\big) = \big(f(t),v\big), \ \forall v \in V \\ u(0) = u_0, \end{aligned}\right\}(3)$$

Where $V = H_0^1(\Omega)$.

We write the semi-discrete (continuous in time) approximation of the advection-diffusion initial boundary value problem (3)

$$\left.\begin{aligned} &\frac{d}{dt}(u_h(t),v_h) + a(u_h(t),v_h) = (f(t),v_h), \\ &\forall v_h \in V_h,\ t \in (0,T) \\ &\qquad u_h(0) = u_{0,h}. \end{aligned}\right\}(4)$$

Here $V_h \subset H_0^1(\Omega)$ is a suitable finite-dimensional space and $u_{0,h} \in V_h$ is an approximation of the initial datum u_0.

II. A CHARACTERISTIC GALERKIN METHOD

We define the characteristic lines associated to a vector field $b = b(t,x)$. Being given $x \in \bar{\Omega}$ and $s \in [0,T]$, they are the vector functions

$X = X(t;s,x)$ such that

$$\left.\begin{aligned}\frac{dX}{dt}(t;s,x) &= b\big(t, X(t;s,x)\big), \quad t \in (0,T)\\ X(s;s,x) &= x.\end{aligned}\right\} \quad (5)$$

The existence and uniqueness of the characteristic lines for each choice of s and x hold under suitable assumptions on b, for instance b continuous in $[0,T]\times\bar{\Omega}$ and Lipschitz continuous in $\bar{\Omega}$, uniformly with respect to $t\in[0,T]$.

From a geometric point of view, $X(t;s,x)$ provides the position at time t of a particle which has been driven by the field b and that occupied the position x at the time s. The uniqueness result gives in particular that

$$X\big(t;s,X(s;\tau,x)\big) = X(t;\tau,x) \quad (6)$$

For each $t,s,\tau\in[0,T]$ and $x\in\bar{\Omega}$.

Hence

$$X\big(t;s,X(s;t,x)\big) = X(t;t,x) = x,$$

i.e., for fixed t and s, the inverse function of $x\to X(s;t,x)$ is given by $y\to X(t;s,y)$.

Therefore, we define

$$\left.\begin{aligned}\bar{u}(t,y) &= u\big(t, X(t;0,y)\big)\\ \text{or equivalenty, } u(t,x) &= \bar{u}\big(t, X(0;t,x)\big).\end{aligned}\right\}(7)$$

From (5), it follows that

$$\frac{\partial\bar{u}}{\partial t}(t,y) = \frac{\partial u}{\partial t}\big(t,X(t;0,y)\big) + \sum_{i=1}^{n} D_i u\big(t,X(t;0,y)\big)\frac{dX_i}{dt}(t;0,y)$$
$$= \left(\frac{\partial u}{\partial t} + b\cdot\nabla u\right)\big(t,X(t;0,y)\big). (8)$$

We can rewrite the non-stationary advection-diffusion equation as

$$\frac{\partial\bar{u}}{\partial t} - \varepsilon\overline{\Delta u} + (\overline{\operatorname{div} b} + \bar{a}_0)\bar{u} = \bar{f} \qquad (9) \ \text{In} Q_T$$

The time derivative is approximated by the backward Euler scheme,

$$\frac{\partial\bar{u}}{\partial t}\big(t_{n+1},y\big) \cong \frac{\bar{u}\big(t_{n+1},y\big) - \bar{u}\big(t_n,y\big)}{\Delta t}. (10)$$

If we set $y = X\big(0;t_{n+1},x\big)$, we have

$$\bar{u}\big(t_{n+1},y\big) = \bar{u}\big(t_{n+1},X(0;t_{n+1},x)\big) = u\big(t_{n+1},x\big).$$

From (7), we have

$$\bar{u}\big(t_n,y\big) = u\big(t_n,X(t_n;0,y)\big)$$

$$X\big(t_n;0,y\big) = X\big(t_n;0,X(0;t_{n+1},x)\big) = X\big(t_n;t_{n+1},x\big).$$
(3.32)

Then we obtain

$$\frac{\partial\bar{u}}{\partial t}\big(t_{n+1},X(0;t_{n+1},x)\big) \cong \frac{u\big(t_{n+1},x\big) - u\big(t_n,X(t_n;t_{n+1},x)\big)}{\Delta t}.$$

We denote a suitable approximation of $X\big(t_n;t_{n+1},x\big)$ by $X^n(x)$, $n = 0,1,\ldots,N-1$, us can write the following implicit discretization scheme for problem ( ).

If we set $u^0 = u_0$, then for $n = 0,1,\ldots,N-1$ we solve

$$\frac{u^{n+1} - u^n \circ X^n}{\Delta t} - \varepsilon\Delta u^{n+1} + \big[\operatorname{div} b\big(t_{n+1}\big) + a_0\big]u^{n+1} = f\big(t_{n+1}\big)$$

InΩ . (11)

A boundary condition has to be imposed on $\partial\Omega$. We consider the homogeneous Dirichlet (1) condition $u^{n+1}|_{\partial\Omega} = 0.$

We choose a backward Euler scheme also for discretizing

$$\frac{dX}{dt}\big(t;t_{n+1},x\big) = b\big(t,X(t;t_{n+1},x)\big). (12)$$

$$\int_{t_n}^{t_{n+1}}\frac{dX}{dt}\big(t;t_{n+1},x\big)dt = \int_{t_n}^{t_{n+1}} b\big(t;X(t;t_{n+1},x)\big)dt.$$

This produces the following approximation of $X\big(t_n;t_{n+1},x\big)$:

$$X^n_{(1)}(x) = x - b\big(t_{n+1},x\big)\Delta t. (13)$$

Here $X^n_{(1)}$ is a second order approximation of $X\big(t_n;t_{n+1},x\big)$, since we are integrating (12) on the time interval $\big(t_n,t_{n+1}\big)$, which has length Δt.

A more accurate scheme is provided by the second order Runge-Kutta scheme,

$$X^n_{(2)}(x) = x - b\left(t_{n+1/2}, x - b(t_{n+1},x)\frac{\Delta t}{2}\right)\Delta t, (14)$$
(3.41)

Which gives a third order approximation of $X(t_n;t_{n+1},x)$.

It is necessary to verify that $X^n_{(i)}(x)\in\Omega$ for each $x\in\bar{\Omega}$, $i = 1,2$, so that we can compute

$u^n \circ X^n_{(i)}$. we assume that $b(t,x)=0$ for each $t\in[0,T]$ and $x\in\partial\Omega$.

As a consequence, $X^n_{(i)}(x)=x$ for $x\in\partial\Omega,\ i=1,2$. If we denote by $x^*\in\partial\Omega$ the point having minimal distance from $x\in\Omega$, we have

$$\left|X^n_{(1)}(x)-x\right| = \left|b\left(t_{n+1},x\right)\right|\Delta t,$$

$$= \left|b\left(t_{n+1},x\right)-b\left(t_{n+1},x^*\right)\right|\Delta t,\quad x^*\in\partial\Omega$$

$$\left|X^n_{(1)}(x)-x\right| \le \sup_{\substack{x,x^*\in\partial\Omega\\ x\ne x^*}} \frac{\left|b\left(t_{n+1},x\right)-b\left(t_{n+1},x^*\right)\right|}{\left|x-x^*\right|}\Delta t\left|x-x^*\right|$$

$$\le \left|b\left(t_{n+1}\right)\right|_{Lip(\bar\Omega)}\left|x-x^*\right|\Delta t,$$

Where $\left|g\right|_{Lip(\bar\Omega)} = \sup_{\substack{x_1,x_2\in\bar\Omega\\ x_1\ne x_2}} \frac{\left|g(x_1)-g(x_2)\right|}{\left|x_1-x_2\right|}$.

We assume that $\max_{t\in[0,T]}\left|b(t)\right|_{Lip(\bar\Omega)}\Delta t<1$.(15)

Then, we have

$\left|X^n_{(1)}(x)-x\right| < \left|x-x^*\right|$, For each $n=0,1,\dots,N-1$.

It follows that $X^n_{(1)}(x)\in\Omega$ for each $x\in\Omega$. a similar result holds for $X^n_{(2)}(x)$. if we suppose that

$$\operatorname{div} b(t,x)+a_0(x)\ge 0 \qquad (16)$$

For each $t\in[0,T]$ and almost every $x\in\Omega$, stability is proven. In fact, multiplying (11) by u^{n+1} and integrating over Ω

, we obtain

$$\int_\Omega \frac{\left(u^{n+1}-u^n\circ X^n_{(1)}\right)}{\Delta t}u^{n+1}\,dx-\int_\Omega \varepsilon\,\Delta u^{n+1}u^{n+1}dx$$

$$+\int_\Omega\left[\operatorname{div} b\left(t_{n+1}\right)+a_0\right]\left(u^{n+1}\right)^2dx = \int_\Omega f\left(t_{n+1}\right)u^{n+1}\,dx.$$

By using Gauss-divergence theorem, we have

$$\int_\Omega\left(u^{n+1}\right)^2dx-\int_\Omega\left(u^n\circ X^n_{(1)}\right)u^{n+1}dx+\varepsilon\Delta t\int_\Omega \nabla u^{n+1}\cdot\nabla u^{n+1}dx$$

$$+\Delta t\int_\Omega\left(\operatorname{div} b(t_{n+1})+a_0\right)\left(u^{n+1}\right)^2dx = \Delta t\int_\Omega f\left(t_{n+1}\right)u^{n+1}dx.$$

By using Hölder's inequality, we have

$$\left\|u^{n+1}\right\|_0^2+\varepsilon\Delta t\left\|\nabla u^{n+1}\right\|_0^2 \le \left(\left\|u^n\circ X^n_{(1)}\right\|_0+\Delta t\left\|f\left(t_{n+1}\right)\right\|_0\right)\left\|u^{n+1}\right\|_0.$$

(17)

From (15), it follows that the map $X^n_{(1)}$ is injective. Therefore, we introduce the change of variable $y=X^n_{(1)}(x)$, and setting $Y^n_{(1)}(y)=\left(X^n_{(1)}\right)^{-1}(y)$, we have

$$\int_\Omega\left(u^n\circ X^n_{(1)}\right)^2dx = \int_{X^n_{(1)}(\Omega)}\left(u^n\left(X^n_{(1)}(x)\right)\right)^2dx$$

$$\left\|u^n\circ X^n_{(1)}\right\|_0^2 = \int_{X^n_{(1)}(\Omega)}\left(u^n(y)\right)^2\left|\det\left(\operatorname{Jac}X^n_{(1)}\right)\circ Y^n_{(1)}(y)\right|^{-1}dy.(18)$$

On the other hand, from (13), we have

$$\left|\det\left(\operatorname{Jac}X^n_{(1)}\right)(x)\right| \ge 1-\Delta t\left|\operatorname{Jac}b\left(t_{n+1}\right)\right|$$

$$\ge 1-\Delta tC_1\left\|\operatorname{Jac}b\left(t_{n+1}\right)\right\|_{L^\infty(\Omega)}$$

$$\ge 1-\Delta tC_1\max_{t\in[0,T]}\left\|\operatorname{Jac}b\left(t\right)\right\|_{L^\infty(\Omega)}$$

$$=1-C_1\mu_1^*\Delta t$$

$$\ge 1-C_1C_2>0$$

For almost every $x\in\Omega$, provided that

(3.42)

$\mu_1^*\Delta t\le C_2$, (19)

Where $\mu_1^*=\max_{t\in[0,T]}\left\|\operatorname{Jac}b(t)\right\|_{L^\infty(\Omega)}$ and $C_1>0,\ 0<C_2<C_1^{-1}$ are suitable constants. From (18), we have

$$\left\|u^n\circ X^n_{(1)}\right\|_0^2 \le \int_{X^n_{(1)}(\Omega)}\left(u^n(y)\right)^2\left(1-C_1\mu_1^*\Delta t\right)^{-1}dy$$

$$\left\|u^n\circ X^n_{(1)}\right\|_0^2 \le \left(1+\Delta tC_3\mu_1^*\right)\left\|u^n\right\|_0^2.(20)$$

Therefore condition (19) implies (15) (if C_2 is small enough).

From (17), we finally obtain for each $n=0,1,\dots,N-1$,

$$\left\|u^{n+1}\right\|_0+\varepsilon\Delta t\frac{\left\|\nabla u^{n+1}\right\|_0^2}{\left\|u^{n+1}\right\|_0} \le \left\|u^n\circ X^n_{(1)}\right\|_0+\Delta t\left\|f\left(t_{n+1}\right)\right\|_0.$$

by using Poincarè's inequality, we have

$$\left\|u^{n+1}\right\|_0+\varepsilon\Delta t\left\|\nabla u^{n+1}\right\|_0 \le \left\|u^n\circ X^n_{(1)}\right\|_0+\Delta t\left\|f\left(t_{n+1}\right)\right\|_0$$

$$\left(\left\|u^{n+1}\right\|_0^2+\varepsilon^2\Delta t^2\left\|\nabla u^{n+1}\right\|_0^2\right)^{\frac12} \le \left(1+C_3\mu_1^*\Delta t\right)^{\frac12}\left\|u^n\right\|_0+\Delta t\left\|f\left(t_{n+1}\right)\right\|_0$$

$$\le\left(1+C_3\mu_1^*\Delta t\right)^{\frac{n+1}{2}}\left\|u_0\right\|_0+\Delta t\sum_{k=1}^{n+1}\left(1+C_3\mu_1^*\Delta t\right)^{\frac{n+1-k}{2}}\left\|f\left(t_k\right)\right\|_0$$

$$\le \left(\|u_0\|_0 + t_{n+1}\max_{t\in[0,T]}\|f(t)\|_0\right)\exp\left(\frac{C_3}{2}\mu_1^* t_{n+1}\right).(21)$$

Therefore $L^2(\Omega)$-stability holds independently of ε.

The convergence of u^n to $u(t_n)$ is proven in a similar way. Defining the error function $\grave{o}^n = u(t_n) - u^n$, from (11), we obtain for $n = 0,1,\dots,N-1$

$$\frac{u(t_{n+1}) - \grave{o}^{n+1} - \left(u(t_n)\circ X_{(1)}^n - \grave{o}^n \circ X_{(1)}^n\right)}{\Delta t}$$
$$-\varepsilon\left(\Delta u(t_{n+1}) - \Delta\grave{o}^{n+1}\right)$$
$$+\left(\operatorname{div} b(t_{n+1}) + a_0\right)\left(u(t_{n+1}) - \grave{o}^{n+1}\right) = f(t_{n+1}),$$

$$\frac{u(t_{n+1}) - u(t_n)\circ X_{(1)}^n}{\Delta t} - \varepsilon\Delta u(t_{n+1})$$
$$+\left(\operatorname{div} b(t_{n+1}) + a_0\right)u(t_{n+1}) - f(t_{n+1})$$
$$= \frac{\grave{o}^{n+1} - \grave{o}^n \circ X_{(1)}^n}{\Delta t} - \varepsilon\Delta\grave{o}^{n+1} + \left(\operatorname{div} b(t_{n+1}) + a_0\right)\grave{o}^{n+1}.$$

Since

$$-\frac{\partial\bar{u}}{\partial t}(t_{n+1}) = -\varepsilon\Delta u(t_{n+1}) + \left(\operatorname{div}\ b(t_{n+1}) + a_0\right)u(t_{n+1}) - f(t_{n+1})$$

and

$$-\frac{\partial\bar{u}}{\partial t}(t_{n+1}) = -\left(\frac{\partial u}{\partial t}(t_{n+1}) + b(t_{n+1})\cdot\nabla u(t_{n+1})\right).$$

Then, we obtain

$$\frac{\grave{o}^{n+1} - \grave{o}^n \circ X_{(1)}^n}{\Delta t} - \varepsilon\Delta\grave{o}^{n+1} + \left[\operatorname{div} b(t_{n+1}) + a_0\right]\grave{o}^{n+1}$$

$$= \frac{u(t_{n+1}) - u(t_n)\circ X_{(1)}^n}{\Delta t} - \frac{\partial u}{\partial t}(t_{n+1}) - b(t_{n+1})\cdot\nabla u(t_{n+1})(22)$$

in Ω.

Since

$$X_{(1)}^n(x) - X(t_n; t_{n+1}, x) = O\left((\Delta t)^2\right),$$

it shows that the right hand side in (22) is $O(\Delta t)$. Therefore convergence follows from (21) applied to $\grave{o}^n$, recalling that $\grave{o}^0 = 0$.

III. CONCLUSION

This paper has presented semi- discrete approximation of the parabolic problem. And the stability and convergency of non- stationary advection- diffusion equation is solved by using characteristic Galerkin method.

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