The Existence of Arbitrarily Tough and Triangle-Free Graphs

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Abstract -- In this paper we mention vertex connectivity and independence number. We establish that every hamiltonian graph and any G1 graph are 1-tough. And then, we describe the bound of the toughness \( t(G) \) in terms of independence number \( \beta(G) \) and the number of vertices, \( n \) in \( G \). Finally, a 1-tough graph \( G_1 \), it is shown that and the result reveals that a triangle-free graph with are obtained.

Indexed Terms - connectivity, independence number, minimum degree, layers of \( G \), 1-tough, Hamiltonian graph, complete bipartite, triangle-free graph.

I. INTRODUCTION

A graph \( G = (V(G), E(G)) \) with \( n \) vertices and \( m \) edges consists of a vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and an edge set \( E(G) = \{e_1, e_2, \ldots, e_m\} \) where each edge consists of two vertices called its end-vertices. If \( uv \in E(G) \), then \( u \) and \( v \) are adjacent. The ends of an edge are said to be incident with the edge. The number of vertices of \( G \) is called the order of \( G \), is denoted by \( \varnothing(G) \). Two vertices \( u \) and \( v \) of \( G \) are said to be connected if there is a \((u, v)\)-path in \( G \).

A graph is said to be connected if every two of its vertices are connected; otherwise it is disconnected. A graph is simple if it has no loops and no parallel edges. The degree of a vertex \( v \) in \( G \) is the number of edges of \( G \) incident with \( v \), each loop counting as two edges. We denote by \( \delta(G) \) and \( \Delta(G) \) the minimum and maximum degrees, respectively of vertices of \( G \). A complete graph \( G \) is a simple graph in which every pair of vertices is adjacent. If a complete graph \( G \) has \( n \) vertices, then it will be denoted by \( K_n \). A spanning subgraph of \( G \) is a subgraph \( H \) with \( V(H) = V(G) \).

A walk in \( G \) is a finite sequence \( W = v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k \), whose terms are alternately vertices and edges, such that, for \( 1 \leq i \leq k \), the ends of \( e_i \) are \( v_{i-1} \) and \( v_i \).

We say that \( W \) is a walk from \( v_0 \) to \( v_k \) or a \((v_0, v_k)\)-walk. The vertices \( v_0 \) and \( v_k \) are called the origin and terminus of \( W \), respectively and \( v_1, v_2, \ldots, v_{k-1} \) its internal vertices. The integer \( k \) is the length of \( W \). If all the edges of a walk are distinct, then it is called a trail. If, in addition, the vertices are distinct, \( W \) is called a path. Suppose that \( V' \) is a nonempty subset of \( V \). A cycle is a closed trail in which all the vertices are distinct, except that the first vertex equals the last vertex. The component of a graph \( G \) is the maximal connected subgraph of \( G \). We denote the number of components of \( G \) by \( \omega(G) \). The ith neighborhood of \( v \) is \( N_i(v) = \{u \in V \mid d(u, v) = i\} \). We set \( N_0(v) = \{v\} \) and abbreviate \( N_i(v) \) to \( N(v) \) and call it the neighbor of \( v \).

The subgraph of \( G \) whose vertex set is \( V' \) and whose edge set is the set of those edges of \( G \) that have both ends in \( V' \) is called the subgraph of \( G \) induced by \( V' \) and is denoted by \( G[V'] \); we say that \( G[V'] \) is an induced subgraph of \( G \). Now suppose that \( E' \) is a nonempty subset of \( E \). The subgraph of \( G \) whose vertex set is the set of ends of edges in \( E' \) and whose edge set is \( E' \) is called the subgraph of \( G \) induced by \( E' \) and is denoted by \( G[E'] \); \( G[E'] \) is an edge-induced subgraph of \( G \).

The vertex-connectivity or simply the connectivity \( \kappa(G) \) of a graph \( G \) is the minimum cardinality of a vertex-cut of \( G \) if \( G \) is not complete, and \( \kappa(G) = n - 1 \) if \( G = K_n \) for some positive integer \( n \). Hence \( \kappa(G) \) is the minimum number of vertices whose removal results in a disconnected or trivial graph. If \( G \) is either trivial or disconnected, \( \kappa(G) = 0 \). \( G \) is said to be \( k \)-connected if \( \kappa(G) \geq k \). All non-trivial connected graphs are \( 1 \)-connected.
The edge-connectivity $\kappa'(G)$ of a graph $G$ is the minimum cardinality of an edge-cut of $G$ if $G$ is non-trivial, and $\kappa'(K_1) = 0$. So $\kappa'(G)$ is the minimum number of edges whose removal from $G$ results in a disconnected or trivial graph. Thus $\kappa'(G) = 0$ if and only if $G$ is disconnected or trivial; while $\kappa'(G) = 1$ if and only if $G$ is connected. A graph $G$ is $k$-edge-connected, $k \geq 1$, if $\kappa'(G) \geq k$.

A bipartite graph is one whose vertex set can be partitioned into two subsets $X$ and $Y$, so that each edge has one end in $X$ and one end in $Y$; such a partition $(X, Y)$ is called a bipartition of the graph. A complete bipartite graph is a simple bipartite graph with bipartition $(X, Y)$ in which each vertex of $X$ is joined to each vertex of $Y$: if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$.

A path that contains every vertex of $G$ is called a Hamilton path of $G$; similarly, a Hamilton cycle of $G$ is a cycle that contains every vertex of $G$. A graph is hamiltonian if it contains a Hamilton cycle.

II. THE $t$-TOUGH AND TRIANGLE-FREE GRAPH

A parameter that plays an important role in the study of toughness is the independence number. Two vertices that are not adjacent in a graph $G$ are said to be independent. A set $S$ of vertices is independent if every two vertices of $S$ are independent. The vertex independence number or simply the independence number $\beta(G)$ of a graph $G$ is the maximum cardinality among the independent sets of vertices of $G$. Let $F$ be a graph. A graph $G$ is $F$-free if $G$ contains no induced subgraph isomorphic of $F$. A $K_{3,3}$-free graph is also referred to as a claw-free graph. If $G$ is a noncomplete graph and $t$ is a nonnegative real number such that $t \leq \frac{|S|}{\omega(G-S)}$ for every vertex-cut $S$ of $G$, then $G$ is defined to be $t$-tough. If $G$ is a $t$-tough graph and $s$ is a nonnegative real number such that $s < t$, then $G$ is also $s$-tough. The maximum real number $t$ for which a graph $G$ is a $t$-tough is called the toughness of $G$ and is denoted by $t(G)$. Since complete graphs do not contain vertex-cuts, this definition does not apply to such graphs.

Consequently, we define $t(K_n) = +\infty$ for every positive integer $n$. Certainly, the toughness of a noncomplete graph is a rational number. Also $t(G) = 0$ if and only if $G$ is a disconnected. It follows that if $G$ is a noncomplete graph, then

$$t(G) = \min \frac{|S|}{\omega(G-S)},$$

where the minimum is taken over all vertex-cuts $S$ of $G$.

Let $G$ be a graph with vertices $v_1, v_2, \ldots, v_n$, and let $\ell \geq 1$ be an integer. We begin by defining the graph $G_{\ell}$ which is constructed by layering $G\ell$ times. For each $k$, $1 \leq k \leq \ell$, the $k$th layer of $G_\ell$ will induce a complete bipartite graph with bipartition sets $\{u_{k,1}, u_{k,2}, \ldots, u_{k,n}\}$ and $\{w_{k,1}, w_{k,2}, \ldots, w_{k,n}\}$. (See Figure 2, in which the layers of $G_\ell$ are schematically illustrated.) We will denote the set of vertices $\{u_{k,j} / 1 \leq k \leq \ell, 1 \leq j \leq n\}$ as Top, and the remaining set of vertices as Bottom. For $1 \leq j \leq n$, the set of vertices $\{u_{1,j}, u_{2,j}, \ldots, u_{\ell,j}\}$ (respectively, $\{w_{1,j}, w_{2,j}, \ldots, w_{\ell,j}\}$) will be called the $j$th top (respectively $j$th bottom).
2.1 Proposition

If \( G \) is a hamiltonian graph, then for any subset \( S \) of \( V \), \( \omega(G - S) \leq |S| \).

Proof:

For the spanning cycle \( C \) of \( G \) it is true that \( \omega(C - S) \leq |S| \). But \( C \) being a spanning subgraph of \( G \), \( \omega(G - S) \leq \omega(C - S) \). Hence the results.

2.2 Corollary

Every hamiltonian graph is 1-tough.

Proof:

Let \( G \) be a hamiltonian graph. By proposition 2.1, for any subset \( S \) of \( V \), \( \omega(G - S) \leq |S| \). It is also true for any vertex-cut \( S \). Thus,

\[
t(G) = \min \left\{ \frac{|S|}{\omega(G - S)} \right\} \geq 1.
\]

Hence \( G \) is 1-tough.

2.3 Lemma

Let \( G \) be a 1-tough graph on \( n \geq 2 \) vertices, and let \( A, B \subseteq V(G) \) with \( |A| + |B| \geq n + 1 \). Then some vertex in \( A \) is adjacent to a vertex in \( B \).

Proof:

We may assume \( A \cap B \) is an independent set in \( G \), since otherwise we are done. Let \( s = |A \cap B| \geq 1, a = |A - B|, b = |B - A|, \) and \( c = |V(G) - (A \cup B)| \).

By assumption, \( |A| + |B| = (a + s) + (b + s) \geq n + 1 \), and so \( a + b + s \geq n - s + 1 \). But \( a + b + c + s = n \), and thus \( c \leq s - 1 \).

If a vertex in \( A \cap B \) has a neighbor in \( A \cup B \), we would be done, and thus we may assume \( N(A \cap B) \subseteq V(G) - (A \cup B) \) and \( c \geq 1 \). Setting \( X = V(G) - (A \cup B) \), we have that \( \omega(G - X) \geq |A - B| = s \geq 2 \), while \( |X| = c \leq s - 1 \). This contradicts the assumption that \( G \) is 1-tough.

2.4 Lemma

Let \( G \) be a connected graph on \( n \geq 2 \) vertices. Suppose we obtain \( G_\ell \) by Layering \( G \) \( \ell \) times. Then \( G_\ell \) is 1-tough.

Proof:

Let \( X \subseteq V(G_\ell) \) such that \( \omega(G_\ell - X) > 1 \) and \( t(G_\ell) = \frac{|X|}{\omega(G_\ell - X)} \). Let \( V_i \) denote the vertices in the \( i \)th layer of \( G_\ell \), \( X_i = X \cap V_i \), and \( \omega_i = \omega(V_i - X_i) \). If \( X_i = \emptyset \) for some \( i \), then immediately \( G_\ell - X \) is connected, contradicting \( \omega(G_\ell - X) > 1 \). Hence we may assume \( |X_i| \geq 1 \) for all \( i \). Since \( \langle V_i \rangle = K_{n,n} \) is 1-tough, we have \( \omega_i \leq |X_i| \) for all \( i \). But then

\[
|X| = \sum_{i=1}^{\ell} |X_i| \geq \sum_{i=1}^{\ell} |\omega_i| \geq \omega(G_\ell - X),
\]

and thus \( G_\ell \) is 1-tough.
2.5 Theorem

Let $G$ be a 1-tough graph on $n \geq 2$ vertices. Form $G^\ell$ by layering $G$ $\ell$ times. Let $n_i = |V(G_i)|$ and $\beta_i = \beta(G_i)$. Then $t(G_i) \geq \frac{n_i}{\sqrt{2\beta_i}}$.

Proof:

Set $t = t(G_i)$ and assume $t < \frac{n_i}{\sqrt{2\beta_i}}$.

Let $X \subseteq V(G_i)$ such that $\omega(G_i - X) > 1$ and $t = \frac{|X|}{\omega(G_i - X)}$. By Lemma 2.4 we can assume $t \geq 1$.

Let $V_k$ denote the vertices in the $k$th layer of $G_i$ and let $X_k = X \cap V_k$. We assume that $X_k$ has the minimum number of vertices among the $X_k$.

Since $\sum_{k=1}^{\ell} |X_k| = |X| = t\omega(G_i - X)$,

we have $\frac{|X_k|}{\omega(G_i - X)} \leq \frac{t}{\ell}$.

Claim 1. $|X_k|$ satisfies $1 \leq |X_k| \leq \frac{n}{t} \leq n$.

Proof of Claim 1. If $|X_k| = 0$, then obviously $\omega(G_i - X) = 1$, a contradiction. So we have $|X_k| \geq 1$.

If $|X_k| \geq \frac{n}{t}$, then by (1) we have

Thus $t^2 \geq \frac{\ell n_i}{\beta_i} = \frac{n_i}{2\beta_i}$, and hence $t \geq \frac{n_i}{\sqrt{2\beta_i}}$.

contradicting the assumption. Since $t \geq 1$, we have $n \leq \frac{n}{t}$.

This proves Claim 1.

Since $|X_k| < n$ clearly $V_i - X_i$ contains vertices from both Top and Bottom, and the vertices in $V_i - X_i$ and belong to a single component of $G_i - X$. Henceforth, we will denote this component as $H$.

Let us now partition the layer numbers $\{1, 2, \ldots, \ell\}$ into two sets Small and Big as follows: for $1 \leq j \leq \ell$ (respectively, $j \in \text{Small}$ (respectively, $j \in \text{Big}$) if $|X_j| \leq n - 1$ (respectively, $|X_j| \geq n$). Note that $1 \in \text{Small}$ by Claim 1.

Claim 2. $\bigcup_{j \in \text{Small}} (V_j - X_j)$ (i.e. the vertices which remain in the small layers when $X$ is removed) all belong to the component $H$.

Proof of Claim 2. If $j \in \text{Small} - \{1\}$, then certainly all vertices in $V_j - X_j$ will belong to the same component of $G_i - X$, since $|V_j - X_j| = 2n - |X_j| \geq n + 1$, and so $V_j - X_j$ contains vertices from both Top and Bottom. Thus it suffices to show there is an edge between $V_i - X_i$ and $V_j - X_j$.

Since $V_j - X_j \geq n + 1$ and $V_i - X_i \geq n + 1$ we have

$|V_j - X_j| \cap \text{Top}| + |V_i - X_i| \cap \text{Top}|

+ |V_j - X_j| \cap \text{Bottom}| + |V_i - X_i| \cap \text{Bottom}|

= |V_j - X_j| + |V_i - X_i| \geq 2n + 2$.

so either $|V_i - X_i| \cap \text{Top}| + |V_j - X_j| \cap \text{Top}| \geq n + 1$ or

$|V_j - X_j| \cap \text{Bottom}| + |V_i - X_i| \cap \text{Bottom}| \geq n + 1$. 


Let us assume the former and define
\[ A = (V_1 - X_1) \cap \text{Top} \text{ and } B = (V_1 - X_1) \cap \text{Top}. \] Thus we have \(|A| + |B| \geq n + 1\). Since \(G\) is 1-tough, it follows by Lemma 2.3 (thinking of \(A, B\) as subsets of \(V(G)\)) that some vertex in \(A\) is adjacent to some vertex in \(B\).

This proves Claim 2.

We now know that all vertices in \(\bigcup_{j \in \text{Small}} (V_1 - X_1)\) belong to the component \(H\) of \(G_{n1} - X\).

Let us now turn to a consideration of the layers whose indices are in \(\text{Big}\), recalling that \(j \in \text{Big}\) implies \(X_n \geq n + 1\). We know
\[ t = \frac{|X|}{\omega(G_{n1} - X)} \geq \frac{|X| + |\bigcup_{j \in \text{Big}} X_j|}{\omega(G_{n1} - X)} \geq \frac{|X| + |\text{Big}| n}{\omega(G_{n1} - X)}. \]

However, if \(k \in \text{Big}\), then by Lemma 2.4 the maximum number of vertices (and hence components) in \(V_{n1} - X_{n1}\) that lie outside \(H\) is \(|X_{n1}|\).

Hence \(\omega(G_{n1} - X) \leq 1 + |\text{Big}| |X_{n1}|\), which gives
\[ t \geq \frac{(|X| + |\text{Big}| n)}{(1 + |\text{Big}| |X_{n1}|)}. \]

Since \(n > t |X_{n1}|\) by Claim 1, we conclude
\[ |\text{Big}| \leq \frac{t - |X_{n1}|}{n - t |X_{n1}|}. \] (2)

If \(|X_{n1}| \geq t\), then by (2) we have \(|\text{Big}| \leq 0\), so we have only the component \(H\) in \(G_{n1} - X\), contradicting \(\omega(G_{n1} - X) > 1\). So we can assume \(|X_{n1}| < t\).

If \(n > t^2\) (equivalently \(\frac{n}{t} - |X_{n1}| > t - |X_{n1}|\)), then by (2) we find
\[ |\text{Big}| \leq \frac{1}{t} \frac{t - |X_{n1}|}{\frac{n}{t} - |X_{n1}|} < 1 \leq 1, \]

implying the contradiction \(|\text{Big}| = 0\). But if \(n \leq t^2\), then since \(\beta_n \leq \ell \beta(G)\) and \(t < \sqrt{\frac{n}{2\beta_n}}\), we obtain
\[ n \leq t^2 < \frac{2\ell n}{2\beta_n} \leq \frac{2\ell n}{2\beta(G)} = \frac{n}{\beta(G)} \]

a contradiction since \(\beta(G) \geq 1\).

This completes the proof of Theorem 2.5.

2.6 Lemma \([5]\)

Let \(G\) be a graph with \(n\) vertices and independence number \(\beta\). Let \(G_{\ell n}\) be constructed by layering \(G\) \(\ell\) times. Then \(\beta(G_{\ell n}) \leq 2n + (\ell - 2)\beta\).

2.7 Corollary

Let \(G\) be a 1-tough graph on \(n \geq 2\) vertices with independence number \(\beta\). Form \(G_{\ell n}\) by layering \(G\) \(\ell \leq 2(\frac{n}{\beta} + 1)\) times. Then \(t(G_{\ell n}) \geq (\frac{1}{2})^{\sqrt{\ell}}\).

Proof:

By Lemma 2.6, we have \(\beta(G_{\ell n}) \leq 2n + (\ell - 2)\beta\). Since \(\ell \leq 2(\frac{n}{\beta} + 1)\), this gives \(\beta(G_{\ell n}) \leq 4n\). Now use Theorem 2.5 and \(|V(G_{\ell n})| = n_{\ell n} = 2n\) to obtain
\[ t(G_{\ell n}) \geq (\frac{1}{2})^{\sqrt{\ell}}. \]

2.8 Lemma

Let \(G\) be a triangle-free graph. Then \(\beta(G) + \kappa(G) \geq 2\delta(G)\).

Proof:

This is obvious if \(\kappa(G) = \delta(G)\), since \(\beta(G) \geq \delta(G)\). If \(\kappa(G) < \delta(G)\), let \(X\) be a vertex - cut of cardinality \(\kappa(G)\) and let \(G_1, G_2\) be two of the components of \(G - X\). Since every vertex outside \(X\) has a neighbor outside \(X\), every component of \(G - X\) has an edge. Consider the endvertices of edge \(vw\) in \(G_1\). Since
they have no common neighbor in X, one of them
must have at most $(\frac{1}{2})\kappa(G)$ neighbors in X. Hence
\[
\beta(G) \geq \delta(G) - (\frac{1}{2})\kappa(G),
\]
implies
\[
\beta(G) \geq \beta(G_1) + \beta(G_2) \geq 2\delta(G) - \kappa(G).
\]

III. CONCLUSION

We conclude that a connected graph G have layering
\(G_l\), must be 1-tough. And then a 1-tough graph \(G_l\) have
the toughness
\[
t(G_l) \geq \frac{n_c}{2\beta_l}.
\]
Finally we discuss a triangle-free graph \(G\), which has the bound of
minimum degree \(\delta(G)\) in terms of independence
number \(\beta(G)\) and connectivity \(\kappa(G)\).

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