Toughness in a Cubic Graph

SAN SAN TINT¹, KHAING KHAING SOE WAI²

¹, ² Department of Engineering Mathematics, Technological University (Myitkyina), Myitkyina, Myanmar

Abstract- In this paper we mention vertex-cut and edge-cut. We establish connectivity, edge-connectivity and minimum degree. And then, we discuss toughness \( t(G) \) and independence number \( \beta(G) \) of a graph. Finally the result reveals that the cubic \( G_m = G_1 \) with \( m = \frac{4}{3} \) toughness is obtained.

Indexed Terms- connectivity, edge-connectivity, t-tough, toughness, k-cube, cubic graph, coloring number, vertex-cut, independence number, minimum degree

I. INTRODUCTION

A graph with \( n \) vertices and \( m \) edges consists of a vertex set and an edge set where each edge consists of two vertices called its end-vertices. Two vertices \( u \) and \( v \) of \( G \) are said to be connected if there is a path in \( G \). A graph is said to be connected if every two of its vertices are connected; otherwise it is disconnected. A graph is simple if it has no loops and no parallel edges. The degree of a vertex \( v \) in \( G \) is the number of edges of \( G \) incident with \( v \), each loop counting as two edges. We denote by \( d(G) \) and \( D(G) \) the minimum and maximum degrees, respectively of vertices of \( G \).

Two simple graphs \( G \) and \( H \) are isomorphic (written if and only if there is a bijection such that if and only if a complete graph \( G \) is a simple graph in which every pair of vertices is adjacent. If a complete graph \( G \) has \( n \) vertices, then it will be denoted by \( K_n \).

The vertex-connectivity or simply the connectivity \( k(G) \) of a graph \( G \) is the minimum cardinality of a vertex-cut of \( G \) if \( G \) is not complete, and \( k(G) = n - 1 \) if \( G = K_n \) for some positive integer \( n \). Hence \( k(G) \) is the minimum number of vertices whose removal results in a disconnected or trivial graph. If \( G \) is either trivial or disconnected, \( k(G) = 0 \).

G is said to be k-connected if \( k(G) \geq k \). All non-trivial connected graphs are 1-connected.

The edge-connectivity \( k_e(G) \) of a graph \( G \) is the minimum cardinality of an edge-cut of \( G \) if \( G \) is non-trivial, and \( k_e(K_1) = 0 \). So \( k_e(G) \) is the minimum number of edges whose removal from \( G \) results in a disconnected or trivial graph. Thus \( k_e(G) = 0 \) if and only if \( G \) is disconnected or trivial; while \( k_e(G) = 1 \) if and only if \( G \) is connected. A graph \( G \) is \( k \)-edge-connected, \( k \geq 1 \), if \( k_e(G) \geq k \).

A bipartite graph is one whose vertex set can be partitioned into two subsets \( X \) and \( Y \), so that each edge has one end in \( X \) and one end in \( Y \); such a partition \((X, Y)\) is called a bipartition of the graph. A complete bipartite graph is a simple bipartite graph with bipartition \((X, Y)\) in which each vertex of \( X \) is joined to each vertex of \( Y \) : if \( |X| = m \) and \( |Y| = n \), such a graph is denoted by \( K_{m,n} \).

II. CONNECTIVITY WITH TOUGHNESS

A parameter that plays an important role in the study of toughness is the independence number. Two vertices that are not adjacent in a graph \( G \) are said to be independent. A set \( S \) of vertices is independent if every two of its vertices are independent. The vertex independence number or simply the independence number \( \beta(G) \) of a graph \( G \) is the maximum cardinality among the independent sets of vertices of \( G \).

Let \( F \) be a graph. A graph \( G \) is \( F \)-free if \( G \) contains no induced subgraph isomorphic of \( F \). A \( K_{1,3} \)-free graph is also referred to as a claw-free graph. If \( G \) is a noncomplete graph and \( t \) is a nonnegative real number such that \( t \leq \frac{|S|}{\omega(G - S)} \) for every vertex-cut \( S \) of \( G \), then \( G \) is defined to be \( t \)-tough. If \( G \) is a \( t \)-tough graph and \( s \) is a nonnegative real number such that \( s < t \), then \( G \) is also \( s \)-tough. The maximum real number \( t \) for which a graph \( G \) is a \( t \)-tough is called the toughness of \( G \) and is denoted by \( t(G) \). Since
complete graphs do not contain vertex-cuts, this definition does not apply to such graphs. Consequently, we define \( t(K_n) = +\infty \) for every positive integer \( n \). Certainly, the toughness of a noncomplete graph is a rational number. Also \( t(G) = 0 \) if and only if \( G \) is a disconnected. It follows that if \( G \) is a noncomplete graph, then
\[
t(G) = \min_{S \subseteq V(G)} \frac{|S|}{\omega(G - S)}.
\]

Where the minimum is taken over all vertex-cuts \( S \) of \( G \). Determining the toughness of a graph usually involves some experimentation. The goal is to find a vertex-cut \( S \) that minimizes \( \frac{|S|}{\omega(G - S)} \).

Figure 1 A graph \( G \) of toughness \( \frac{6}{5} \).

For the graph \( G \) of figure 1, \( S_1 = \{g, h, c, d, j, k, o, n\} \), \( S_2 = \{a, c, e, g, q, m\} \), \( S_3 = \{a, d, f, h, l, m, q, p\} \).
\[
\frac{|S_1|}{\omega(G - S_1)} = \frac{8}{6}, \quad \frac{|S_2|}{\omega(G - S_2)} = \frac{6}{5}, \quad \frac{|S_3|}{\omega(G - S_3)} = \frac{8}{5}.
\]

2.1 Theorem
Let \( G \) be a connected graph of order \( n \geq 3 \) that is not complete. For each edge-cut \( X \) of \( G \), there is a vertex-cut \( U \) of \( G \) such that \( |U| \leq |X| \).

Proof: Assume, without loss of generality, that \( X \) is a minimum edge-cut of \( G \). Then \( G - X \) is a disconnected graph containing exactly two components \( G_1 \) and \( G_2 \), where \( G_i \) has order \( n_i \) (\( i = 1, 2 \)). Thus \( n_1 + n_2 = n \). We consider two cases.

Case 1.
Every vertex of \( G_i \) is adjacent to every vertex of \( G_j \). Then \(|X| = n_1n_2\). Since \((n_1 - 1)(n_2 - 1) \geq 0\), it follows that \( n_1n_2 \geq n_1 + n_2 - 1 = n - 1 \) and so \(|X| \geq n - 1\). Since \( G \) is not complete, \( G \) contains two nonadjacent vertices \( u \) and \( v \). Then \( U = V(G) - \{u, v\} \) is a vertex-cut of cardinality \( n - 2 \) and \(|U| < |X|\).

Case 2.
There are vertices \( u \) in \( G_1 \) and \( v \) in \( G_2 \) that are not adjacent in \( G \). For each edge \( e \) in \( X \), we select a vertex for \( U \) in the following way. If \( u \) is incident with \( e \), then choose the other vertex (in \( G_2 \)) incident with \( e \) for \( U \); otherwise, select for \( U \) the vertex that is incident with \( e \) and belongs to \( G_1 \). Now \(|U| \leq |X|\).

Furthermore, \( u, v \in V(G - U) \), but \( G - U \) contains \((u, v)\)-path, so \( U \) is a vertex-cut.

2.2 Theorem
For every graph \( G \), \( \kappa(G) \leq \kappa'(G) \leq \delta(G) \).

Proof: If \( G \) is trivial or disconnected, then \( \kappa(G) = \kappa'(G) = 0 \); so we can assume that \( G \) is a nontrivial connected graph. Let \( v \) be a vertex of \( G \) such that \( \deg v = \delta(G) \), the removal of the \( \delta(G) \) edges of \( G \) incident with \( v \) results in a graph \( G' \) in which \( v \) is isolated, so \( G' \) is either disconnected or trivial. Therefore, \( \kappa'(G) \leq \delta(G) \).

We now verify the other inequality. If \( G = K_n \) for some positive integer \( n \), then \( \kappa(K_n) = \kappa'(K_n) = n - 1 \). Suppose next that \( G \) is not complete, and let \( X \) be an edge-cut such that \(|X| = \kappa'(G)\).

By Theorem 2.1 there exists a vertex-cut \( U \) such that \(|U| \leq |X|\). Thus \( \kappa(G) \leq |U| \leq |X| = \kappa'(G) \).

2.3 Corollary
Let \( G \) be a graph with vertices \( x_1, x_2, \ldots, x_n \), \( d(x_1) \leq d(x_2) \leq \ldots \leq d(x_n) \). suppose
for some \( k, \ 0 \leq k \leq n, \) that \( d(x_j) \geq j + k - 1, \) for \( j = 1, 2, ..., n - 1 - d(x_{n-k+1}), \) then \( G \) is \( k \)-connected.

**Proof**

Suppose that \( G \) is not \( k \)-connected. Then there exist \( V_1, V_2 \subseteq V(G), \) such that \( V_1 \cap V_2 = \emptyset, \) \( |V_1| = n_1, |V_2| = n_2, n_1 + n_2 = n - k + 1 \) and \( d(x) \leq n_i + k - 2 \) for \( x \in V_i. \) Now, \( X = \{ x \mid j \geq n - k + 1 \} \) is a set of \( k \) elements all with a degree larger than or equal to \( d(x_{n-k+1}). \) Hence, there is at least one \( x \in X \cap (V_1 \cup V_2). \) Without loss of generality, say in \( x \in V_1. \) Thus \( n_2 \geq d(x_{n-k+1}) + 1 - (k - 1) = d(x_{n-k+1}) - k + 2 \) and \( n_1 = n - k + 1 - n_2 \leq n - 1 - d(x_{n-k+1}). \) Take \( x_j \in V_1 \) such that \( j \) is maximal \( (j \geq n_j), \) then \( n_i + k - 1 \leq d(x_j) \leq d(x_j) \leq n_1 + k - 2. \)

Thus, if \( G \) is a graph with vertices \( x_1, x_2, ..., x_n, \) with \( d(x_j) \leq d(x_j) \leq ... \leq d(x_n) = \Delta(G) \) and \( d(x_j) \geq j \) for \( j = 1, 2, ..., n - \Delta(G) - 1, \) then \( G \) is connected. The reverse is, obviously, not true.

### 2.4 Corollary

Let \( G \neq K_n \) be a graph of order \( n, \) then \( \kappa(G) \geq 2\delta(G) + 2 - n. \)

**Proof:**

Let \( k = 2\delta(G) + 2 - n. \) It suffices to show \( d(x_i) \geq j + k - 1, \) for \( j = 1, ..., n - 1 - \delta(G) \) (because \( d(x_{n-k+1}) \geq \delta(G) \)). This is certainly true if \( d(x_j) \geq n - 1 - \delta(G) + k - 1 \) for all \( j = 1, ..., n - 1 - \delta(G) \) and \( n - 1 - \delta(G) + k - 1 = \delta(G). \)

### 2.5 Theorem

For every noncomplete graph \( G, \)

\[
\frac{\kappa(G)}{\beta(G)} \leq t(G) \leq \frac{\kappa(G)}{2}.
\]

**Proof:**

The independence number is related to toughness in the sense that among all the vertex-cuts \( S \) of the noncomplete graph \( G, \) the maximum value of \( \omega(G - S) \) is \( \beta(G), \) so for every vertex-cut \( S \) of \( G, \) we have that \( \kappa(G) \leq |S| \) and \( \omega(G - S) \leq \beta(G). \)

By the definition of \( t(G), \)

\[
t(G) = \min \left\{ \frac{|S|}{\omega(G - S)} : \kappa(G) \leq |S| \right\} \geq \frac{\kappa(G)}{\beta(G)}.
\]

Let \( S' \) be a vertex-cut with \( |S'| = \kappa(G). \)

Thus \( \omega(G - S') \geq 2, \) so

\[
t(G) = \min \left\{ \frac{|S|}{\omega(G - S)} : \kappa(G) \leq |S| \right\} \leq \frac{\kappa(G)}{2}.
\]

### 2.6 Theorem [3]

A graph \( G \) of order \( n \geq 2 \) is \( k \)-connected \((1 \leq k \leq n - 1)\) if and only if for each pair \( u, v \) of distinct vertices there are at least \( k \) internally-disjoint \((u, v)\) paths in \( G. \)

### 2.7 Theorem

If \( G \) is a noncomplete claw-free graph, then \( t(G) = \frac{1}{2} \kappa(G). \)

**Proof:**

If \( G \) is disconnected, then \( t(G) = \kappa(G) = 0 \) and the result follows. So we assume that \( \kappa(G) = r \geq 1. \) Let \( S \) be a vertex-cut such that \( t(G) = \frac{|S|}{\omega(G - S)} \), suppose that \( \omega(G - S) = k \) and that \( G_1, G_2, ..., G_k \) are the components of \( G - S. \)

Let \( v_i \in V(G_i) \) and \( u_j \in V(G_j), \) where \( i \neq j. \) Since \( G \) is \( r \)-connected, it follows by Theorem 2.6 that \( G \) contains at least \( r \) internally disjoint \((u_i, u_j)\)-paths. Each of these paths contains a vertex of \( S. \) Consequently, there are at least \( r \) edges joining the vertices of \( S \) and the vertices of \( G_i \) for each \( i \) \((1 \leq i \leq k)\) such that no two of these edges are incident with the same vertex of \( S. \)

Hence there is a set \( X \) containing at least \( kr \) edges between \( S \) and \( G - S \) such that any two edges incident with a vertex of \( S \) are incident with vertices in distinct components of \( G - S. \) However, since \( G \) is claw-free, no vertex of \( S \) is joined to vertices in three components of \( G - S. \) Therefore,

\[
kr = |X| \leq 2|S| = 2k t(G).
\]
So \( kr \leq 2k t(G) \).

Thus \( t(G) \geq \frac{r}{2} = \frac{1}{2} \kappa(G) \).

By Theorem 2.5, \( t(G) = \frac{1}{2} \kappa(G) \).

### III. TOUGHNESS AND INDEPENDENCE NUMBER

In this paper we derive upper bounds on the toughness of cubic graphs in terms of the independence number and coloring parameters.

A coloring of a graph G we mean an assignment of colors to the vertices of G such that any two vertices joined by an edge receive different colors. A color class A is minimal if every vertex of A has a neighbor of every other color.

We show an example of a graph coloring, thus need three colors to color.

A graph is a k-colorable if there is a vertex coloring with k colors. Let be a graph. The chromatic number of G, written is the minimum integer k such that G is k-colorable.

The k-cube is the graph whose vertices are the ordered k-tuples of 0's and 1's, two vertices begin joined if and only if they differ in exactly one coordinate.

![Figure 2. A graph coloring.](image)

A graph coloring is 3-cube. Let \( V \) be the vertex set of k-cube. Let \( v \in V \). Since \( v \) is k-tuple, there are \( k \) edges that differ in \( v \) in exactly one coordinate. There are \( k \)-edges between \( v \) and these vertices.

Let \( E_v \) be the set of edges incident with \( v \).

\[ |E_v| = k \]

For all \( v \in V \), we have the sum

\[ \sum_{v \in V} |E_v| = k(2^k) \]

But in this sum edge is counted twice.

Therefore 2 (number of edges in k-cube) = \( k(2^k) \).

We have seen that the number of edges in k-cube is \( k2^k = k2^{k-1} \).

Let \( X = \{ v \in V \mid \text{the number of is in } v \text{ is odd} \} \).

Let \( Y = \{ v \in V \mid \text{the number of is in } v \text{ is even} \} \).

Take any edge \( uv \) in k-tuple.

Assume that \( u \in X \).
Since \( u \) and \( v \) differ in exactly one coordinate, \( v \in Y \). Thus for every edge \( uv \) in the \( k \)-cube, one end in \( X \) and the other end in \( Y \).
Hence \( k \)-cube is bipartite.

3.2 Lemma [5]
Let \( G \) be a cubic graph with a 3-coloring \((A, B, C)\) such that \( A \) is minimal. If |
\[
\begin{aligned}
|A| &= a, \\
|B| &= b, \\
|C| &= c
\end{aligned}
\]
then
\[
t(G) \leq \frac{3b + a - c}{2b}.
\]

3.3 Theorem
For a noncomplete cubic graph \( G \) on \( n \) vertices and independence number \( \beta \):
\[
t(G) \leq \min \left\{ \frac{2n - 3\beta}{n - \beta}, \frac{2\beta}{4\beta - n} \right\}.
\]
Proof:
By the definition of a graph coloring, a graph of maximum degree \( r \) has an \( r \)-coloring where one of the color classes is a maximum independent set. Let \((A, B, C)\) be a 3-coloring of \( G \) where \( C \) is a maximum independent set, and subject to this, \( A \) is as small as possible. So \( |C| = \beta \) and \( b \geq \frac{(n - \beta)}{2} \).
Also \( 3b \geq 3c - a = 3\beta - (n - b - \beta) \) by inequality, whence \( b \geq \frac{2\beta - n}{2} \).
So by the above Lemma 3.2,
\[
t(G) \leq \frac{3b + a - c}{2b} = 1 + \frac{n - 2\beta}{2b} \leq \min \left\{ \frac{2n - 3\beta}{n - \beta}, \frac{2\beta}{4\beta - n} \right\},
\]
as required.

This is best possible. For \( \beta = \frac{n}{2} \), the theorem gives an upper bound of 1, and any 3-connected cubic bipartite graph has toughness 1. At the other extreme the theorem shows that \( t(G) = \frac{3}{2} \) in noncomplete cubic graphs requires that \( \beta = \frac{n}{3} \) and thus \( n \) a multiple of 3.

Consider also the following cubic graph \( G_m \) for \( m \) a positive integer. Start with a set \( U = \{u_1, \ldots, u_{6m}\} \) of 6m vertices which form a cycle. Then add a set \( V = \{v_1, \ldots, v_m\} \) 3m vertices, and connect \( v_i \) to \( u_{2i-1} \) and \( u_{2i} \). Finally add a set \( W = \{w_1, \ldots, w_m\} \) of \( m \) vertices, and join \( w_i \) to \( v_{3i-2} \), \( v_{3i} \) and \( v_{3i+2} \). The graph is illustrated in Figure 1. The graph \( G_m \) has \( n = 10m \) vertices, toughness \( \frac{4}{3} \) and independence number \( \beta = \frac{2n}{5} \).

The graph is illustrate for the cubic graph \( G_m = G_1 \).

3.4 Proposition [5]
For a noncomplete graph \( G \), \( \kappa(G) \leq t(G) \leq \frac{\kappa(G)}{2} \).

3.5 Lemma
For \( m \geq 1 \), \( G_m \) has toughness \( \frac{4}{3} \).
Proof:
Let \( S \) be a vertex-cut and suppose \( G_m - S \) has \( k \) components. Note that by Proposition 3.4, \( k \leq |S| \) (since \( G_m \) is 3-connected). Thus as \( \frac{|S| + 1}{(k + 1)} \leq \frac{|S|}{k} \) we may assume that \( G_m - S \) has no vertex-cut.
We show first that we may assume that \( S \cap V = \emptyset \). Suppose for some \( W_i \) some of its neighbors are in \( S \).
Then in $G_m - S$ reinsert the neighbors of $w_i$ and remove $w_i$ instead. This action cannot join two components (since $v_j$'s two neighbors in $U$ are already adjacent). The only way this action can reduce the number of components is if $w_i$ a singleton component in was $G_m - S$. But that means all of $w_i$’s neighbors are in $S$ and since 
\[
\frac{|S|-2}{k-1} < \frac{|S|}{k}
\]
we are better off, a contradiction. So this action does not decrease the number of components and does not increase the number of vertices removed. Hence we may assume that $S \cap V = \emptyset$.

Let $w$ denote $|S \cap W|$. Since $G_m - S$ has no cut-vertex, for any $w_i$ not in $S$ the vertices $u_{6i-4}, u_{6i-3}, ..., u_{6i+3}$ must all lie in the component with $w_i$. Denote the subpath $u_{6i-4}, u_{6i-3}, ..., u_{6i+3}$ by $P_i$. Now it is not hard to see that the best strategy, once $S \setminus W$ is determined, is to remove every alternate vertex of $U$ that lies outside the $P_i$ corresponding to the $w_i \not\in S$. The number $u$ of vertices of $U$ removed is equal to the number of components that remain. Also $u \leq 3w$. Hence
\[
t \geq \frac{(w + 3w)}{3w} = \frac{4}{3}.
\]

IV. CONCLUSION

We conclude that connected graph have connectivity and edge-connectivity. The bounds of connectivity of a graph $G$ are expressed as in terms of minimum degree of $G$ and numbers of vertices in $G$. The toughness $t(G)$ is discussed which is related to the connectivity and independence number in $G$. And then the result of the cubic graph $G_m$ has $4/3$ toughness.

REFERENCES


